$\zeta(3)$ is Irrational

A Proof based on Irrationality Criterion

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Overview

Idea of the Proof I

We generate two sequences:

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 c_{n,k}. \tag{1}$$

We use the following notation to abbreviate $\{a_n\}$, i.e., the first sequence:

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}}. \tag{2}$$

and

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2. \tag{3}$$

Idea of the proof II.

Then we have

$$a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3} \tag{4}$$

where a_n, and b_n are defined as (1), and (3).

It follows that

$$\det\left(\begin{bmatrix} a_n & a_{n-1} \\ b_n & b_{n-1} \end{bmatrix}\right) = a_n b_{n-1} - a_{n-1} b_n = \frac{6}{n^3},$$

$$\frac{a_n}{b_n} \longrightarrow \zeta(3) \text{ as } n \longrightarrow \infty.$$

$$\Rightarrow \zeta(3) - \frac{a_n}{b_n} = \sum_{k=n+1}^{\infty} \frac{6}{k^3 b_k b_{k-1}}. \tag{5}$$

Idea of the proof III.

The ratio of the two sequences converges to $\zeta(3)$, then our goal is to modify the two sequences to satisfy the irrationality criterion.

Irrationality Criterion

Corollary. If there exists $\delta > 0$ and infinite pairs $p_n, q_n \in \mathbb{Z}$ and $(p_n, q_n) = 1$ such that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}, \tag{6}$$

then $\alpha \in \mathbb{R}$ is irrational.

Actually, it's an if and only if, but for our goal, we only need one direction.

Idea of the proof IV.

Need to show $\delta > 0$.
Next, define two new sequences: $p_n = 2d_n^3 a_n, q_n = 2d_n^3 b_n$ where $b_n \sim (1+\sqrt{2})^{4n} := \alpha^n$, $p_n, q_n \in \mathbb{Z}$ If we don't define p_n, q_n, and we only use:

$$\zeta(3) - \frac{a_n}{b_n} = O\left(\frac{1}{b_n^2}\right) \text{, then by IC } \Rightarrow \frac{1}{b_n^2} < \frac{1}{b_n^{1+\delta}}$$

$$\Rightarrow 0 < \delta < 1 \text{, and we are done?}$$

Since a_n is not integer, so we can't directly invoke irrationality criterion (IC). That's why we make it become an integer: $p_n := 2d_n^3 a_n$ where $d_n = lcm(1, 2, 3, ..., n)$. But, we don't want to change the estimation of $\left|\zeta(3) - \frac{a_n}{b_n}\right|$, so, we also did this to b_n, and obtain: $q_n := 2d_n^3 b_n$.

Q: Why we can't just use $O\left(\frac{1}{b_n^2}\right)$ in IC?

Now, we can use the irrationality criterion, we also notice the second reason why we can't directly use $\frac{1}{b_n^{1+\delta}}$ on the r.h.s. of the IC, but $\frac{1}{q_n^{1+\delta}}$ instead: From the criterion, if the following is true then $\zeta(3)$ is irrational:

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| = \left|\zeta(3) - \frac{a_n}{b_n}\right| = O\left(\frac{1}{b_n^2}\right) < \frac{1}{q_n^{1+\delta}} = \frac{1}{(2d_n^3 b_n)^{1+\delta}}$$

$$\Rightarrow \frac{1}{b_n^2} < \frac{1}{(2d_n^3 b_n)^{1+\delta}}$$

It follows: (1) *if we take* $\delta = 1$ **the inequality isn't valid**, (2) $0 < \delta < 1$ which is what we need (and we can do a further estimation to find out the exact maximum value of $\delta = 0.080\ldots$

Idea of the proof V.

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| = \left|\zeta(3) - \frac{a_n}{b_n}\right| = O\left(\frac{1}{b_n^2}\right) = O(\alpha^{-2n}).$$

Take

$$0 < \delta = \frac{\log \alpha - 3}{\log \alpha + 3} = 0.080529\ldots.$$

We obtain

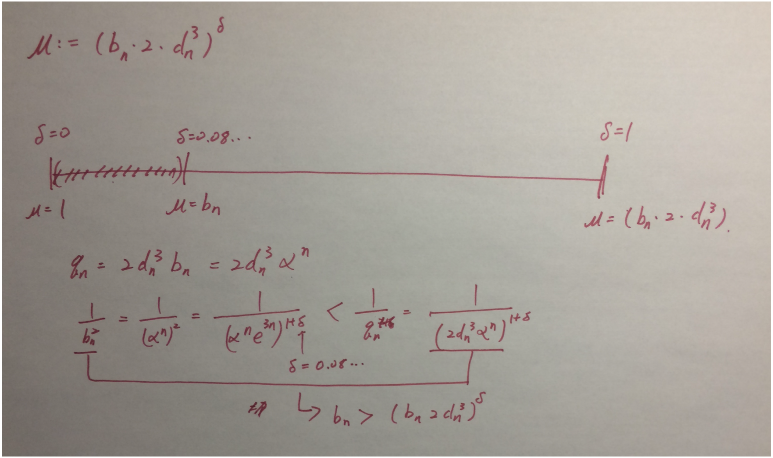
$$\log \alpha = \frac{3(1+\delta)}{1-\delta} \Rightarrow \alpha^{-1+\delta} = e^{-3(1+\delta)}$$

$$\Rightarrow \alpha^{-2}\alpha^{1+\delta} = e^{-3(1+\delta)}$$

When n is large enough, we can have the following inequality:

$$\Rightarrow (\alpha^n 2 d_n^3)^{-(1+\delta)} = \frac{1}{q_n^{1+\delta}} > \alpha^{-2n} = \frac{1}{(\alpha^n e^{3n})^{1+\delta}} = O\left(\frac{1}{b_n^2}\right)$$

Idea of the proof V.



Idea of the proof V.

Therefore, this δ is the maximum of all possible values to make the inequality be true. It follows that

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{1+\delta}}$$

and this implies $\zeta(3)$ is irrational.

References

[1] van der Poorten, Alfred (1979), "A proof that Euler missed ... Apréy's proof of the irrationality of (3)." The Mathematical Intelligencer, 1 (4): 195-203.

[2] A. Ya. Khinchin, (1964), "Continued Fractions." The University of Chicago Press.

Thank you!