Spectrum of a Compact Operator

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Overview

Notations

H represents Hilbert space.
Λ is for bounded linear operators in H.
A stands for linear operator in $\mathbb{R}^n$.
$(,)$ inner products in Hilbert space H.
$\doteq$ means equal by definition.

Definitions.

- Λ is **symmetric** if $(\Lambda x, y) = (x, \Lambda y)$, for all $x, y \in H$, so Λ is **self-adjoint**
- The resolvent set of Λ, denoted as $\rho(\Lambda)$, is the set of numbers $\eta \in \mathbb{R}$ such that $\eta I - \Lambda$ is a bijection[1]
- The complement of the resolvent set: $\sigma(\Lambda) \doteq \mathbb{R} \backslash \rho(\Lambda)$ is called the **spectrum**.
- The **point spectrum** of Λ, denoted as $\sigma_p(\Lambda)$, is the set of numbers $\eta \in \mathbb{R}$ such that $\eta I - \Lambda$ is not injective. In other words, If there exists a nonzero vector $w \in H$ such that

$$\Lambda w = \eta w$$

then $\eta \in \sigma_p(\Lambda)$ where η is an eigenvalue of Λ, and w is and associated **eigenvector**.
- The **essential spectrum** of Λ, denoted as $\sigma_e(\Lambda) = \sigma(\Lambda) \backslash \sigma_p(\Lambda)$, is the set of numbers $\eta \in \mathbb{R}$ such that $(\eta I - \Lambda)$ is injective, not surjective.

[1] By open mapping theorem, $(\eta I - \Lambda)^{-1}$ is continuous.

The chapter 6 is initiated from classical linear algebra. For a linear operator $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ in a finite-dimentional space, there are two results are what we would like to generalize to infinite-dimensional Hilbert space H:

- A is one-to-one if and only if A is onto, since $\dim(Ker(A)) = (Range(A))^{\perp}$.
- If A is symmetric, then its eigenvalues are real, and the space $\mathbb{R}^n$

The first result is still valid for operators with the form:

$$\Lambda = I - K,$$

where I is the identity, and K is a compact operator.
The second statement can be extended to any compact, and self-adjoint operator $\Lambda : H \mapsto H$.

Fredholm Theorem.

Let H be a Hilbert space over the reals and let $K : H \mapsto H$ be a compact linear operator. Then

- Ker$I - K$ is finitie-dimensional
- Range$(I - K)$ is closed
- Range$(I - K)$=Ker$(I - K^*)^{\perp}$
- Ker$(I - K) = \{0\} \Leftrightarrow$ Range$(I - K) = H$
- Ker$(I - K)$ and Ker$(I - K^*)$ have the same direction

This theorem tells us whether a linear equation:

$$u - Ku = f$$

has solutions, and if so, whether those solutions are unique.

Tow cases

- Case 1: $\mathrm{Ker}(I - K) = \{0\}$. The operator $I - K$ is one-to-one and onto. For every $f \in H$ the above linear equation has a unique solution.
- Case 2: $\mathrm{Ker}(I - K) \neq \{0\}$ Hence the homogeneous equation $u - Ku = 0$ has a nontrivial solution. The above linear equation has solutions, if and only if $f \in \mathrm{Ker}(I - K^*)^{\perp}$. That is, if and only it, $(f, u) = 0$, $\forall u \in H$ such that $u - K^*$.

Spectrum of a compact operator

Theorem

Let H be an infinite-dimensional Hilbert space, and let $K : H \mapsto H$ be a compact linear operator. Then

- $0 \in \sigma(K)$
- $\sigma(K) = \sigma_p(K) \cup \{0\}$
- *Either $\sigma_p(K)$ is finite, or else $\sigma_p(K) = \{\lambda_k : k \geq 1\}$, where the eigenvalues satisfy $\lim_{k \mapsto \infty} \lambda_k = 0$*

Bounds on the spectrum of a symmetric operator

Lemma

Let $\Lambda : H \mapsto H$ be a bounded linear selfadjoint operator on a real Hilbert space H. Define the upper and lower bounds

$$m \doteq \inf_{u \in H, \|u\|=1} (\Lambda u, u), \quad M \doteq \sup_{u \in H, \|u\|=1} (\Lambda u, u). \tag{1}$$

Then

- *The spectrum $\sigma(\Lambda)$ is contained in the interval $[m, M]$*
- $m, M \in \sigma(\Lambda)$
- $\|\Lambda\| = \max\{-m, M\}$

References

[1] Alberto Bressan (2013), "Lecture Notes on Functional Analysis–with Applications to Linear Partial Differential Equations." American Mathematical Society.

[2] Anatole Katok (2011), "Spaces: From Analysis to Geometry and Back." Lecture Notes from MASS 2011 course in Analysis.

Thank you!