Quantum Field Theory II.

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3) Study the free field theory $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2$ at finite temperature and derive the Bose-Einstein distribution.

According to (4).

$$\Rightarrow Z = \mathrm{tr}\, e^{-\beta H} = \int_{PBC} Dq\, e^{-\int_0^\beta d\tau L(q)} \quad \text{——(4)}$$

↓ extend to field theory: If H is the Hamiltonian of a quantum field theory in D-dimensional space ($d = D+1$), then (4) becomes

$$Z = \mathrm{tr}\, e^{-\beta H} = \int_{PBC} D\varphi\, e^{-\int_0^\beta d\tau \int d^D x\, \mathcal{L}(\varphi)} \quad \text{——(5)}$$

$$\varphi(\vec{x}, 0) = \varphi(\vec{x}, \beta)$$

Recall for a harmonic oscillator, the action is quadratic in the dynamical variables,

$$S[x] = \int_{t_i}^{t_f} dt \left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right]$$

$$\langle x_f | e^{-\frac{i}{\hbar}HT} | x_i \rangle = \int Dx\, e^{\frac{i}{\hbar}S[x]} = \left(\frac{m\omega}{2\pi i\hbar \sin\omega T}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}S[x_{cl}]}$$

$\Rightarrow$ transition amplitude is given for zero temperature in above eq. $\quad T = t_f - t_i$

Now, making the identifications $T = -i\beta$, $\varphi_i = \varphi_f = \varphi$

$$\Rightarrow Z(\beta) = \int dx \left(\frac{m\omega}{2\pi \sinh\beta\omega}\right)^{\frac{1}{2}} e^{-(m\omega \tanh\frac{\beta\omega}{2})x^2}$$

↓

$$= \left(\frac{m\omega}{2\pi \sinh \beta\omega}\right)^{\frac{1}{2}} \left(\frac{\pi}{m\omega \tanh\left(\frac{\beta\omega}{2}\right)}\right)^{\frac{1}{2}}$$

$$= \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega} - 1} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} = \frac{1}{2\sinh\left(\frac{\hbar\omega}{kT}\right)}$$

Now, consider a more general case (with chemical potential!)

$\Rightarrow$ Assume system that is free to exchange both energy and particle with a reservoir. (grand canonical ensemble and system described by temperature T and chemical potentials μ_i. Density matrix

$$\hat{\rho} \equiv Z^{-1} \exp[-\underbrace{\beta}_{=\frac{1}{T}} (\underbrace{\hat{H} - \mu_i \hat{N}_i}_{\equiv \bar{H}})]$$

$$\langle \hat{A} \rangle = Tr[\hat{\rho}\hat{A}]$$

$\Rightarrow$ Consider one system with $E = \omega$, and ignore zero point energy $\hat{H}|n\rangle = \omega\hat{N}|n\rangle = n\omega|n\rangle$.

Thus, for bosons with chemical potential μ,

$$Z_b = Tr\, e^{-\beta(\omega-\mu)\hat{N}} = \sum_{n=1}^{\infty} \langle n | e^{-\beta(\omega-\mu)\hat{N}} | n\rangle$$

$$= \frac{1}{1 - e^{-\beta(\omega-\mu)}}$$

take derivative

$$N_{boson} = \frac{1}{e^{\beta(\omega-\mu)} - 1}, \quad E_b = N_b\, \omega.$$

for g_i degeneracy

$$\boxed{n_i = \frac{g_i}{e^{\beta(\omega-\mu)} - 1}}$$

ow, I try to generalized to scalar field.

The free Euclidean action in terms of the phase space variables S_0, is in this case given by

$$S_0 = \int \underset{\substack{\uparrow \\ d\tau d^D x}}{d^{d+1}x} \left[-i\pi \partial_\tau \varphi + \mathcal{H}_0(\pi, \varphi) \right]$$

$$\mathcal{H}_0(\pi, \varphi) \equiv \frac{1}{2}\left[\pi^2 + |\nabla\varphi|^2 + m^2\varphi^2\right].$$

recall $\varphi(\beta, \vec{x}) = \varphi(0, \vec{x})$

$\Rightarrow \pi(\beta, \vec{x}) = \pi(0, \vec{x})$, $\forall \vec{x} \in \mathbb{R}^d$.

which requires the introduction of two time-independent Lagrange multiplier fields: $\xi_a(\vec{x})$, $a = 1, 2$.

Defining a two-component field $\Phi = (\Phi_a)_{a=1,2}$

such that $\underbrace{\Phi_1 = \varphi, \ \Phi_2 = \pi.}$

are analogous procedure to harmonic oscillator case in QM.

For free partition function $Z_0(\beta)$:

$$\Rightarrow Z_0(\beta) = N^{-1} \int D\xi \int D\Phi \, e^{\mp\frac{1}{2}\int d^{d+1}x \, \Phi_a \hat{K}_{ab} \Phi_b + i\int d^{d+1}x \, j_a \Phi_a}$$

$$j_a(x) \equiv \xi_a(x)\left[\delta(\tau - \beta) - \delta(\tau)\right]$$

$$\hat{K} = \begin{pmatrix} \hat{h}^2 & i\frac{\partial}{\partial\tau} \\ -i\frac{\partial}{\partial\tau} & 1 \end{pmatrix}, \quad \hat{h} \equiv \sqrt{-\nabla^2 + m^2}$$

[the first-quantized energy operator for massive scalar particles performing the integral over Φ,

$$Z_0(\beta) = \int D\xi \, e^{\frac{-1}{2}\int d^dx \int d^dy \, \xi_a(x) \langle x | \hat{M}_{ab} | y \rangle \xi_b(y)}.$$

$$\hat{M} \equiv \hat{\Omega}(0_+) + \hat{\Omega}(0_-) - \hat{\Omega}(\beta) - \hat{\Omega}(-\beta).$$

and $\hat{\Omega} \equiv \begin{pmatrix} \frac{1}{2}\hat{h}^{-1} & \frac{i}{2}\operatorname{sgn}(\tau) \\ \frac{-i}{2}\operatorname{sgn}(\tau) & \frac{1}{2}\hat{h} \end{pmatrix} e^{-\hat{h}|\tau|}$

$\Rightarrow$ substitute $\hat{\Omega}$ into $\hat{M}$.

we have

$$\hat{M} \equiv \begin{pmatrix} \hat{h}^{-1} & 0 \\ 0 & \hat{h} \end{pmatrix} (\hat{n}_B + 1)^{-1}.$$

after simplified

Where $\hat{n}_B \equiv \dfrac{1}{e^{\beta\hat{h}} - 1}$

Method II.

For simplicity, I rewrite partition function as.

$$Z_0 = \int d[\phi^*] d[\phi] e^{-S_0[\phi^*, \phi]/\hbar}.$$

incorporate the boundary conditions by expanding the fields as $\phi_\alpha(x, \tau) = \sum_{\hat{n}, n} \phi_{\hat{n}, n, \alpha} \chi_{\hat{n}}(x) \dfrac{e^{-i\omega_n \tau}}{\sqrt{\hbar\beta}}$.

$\omega_n = \dfrac{\pi(2n)}{\hbar\beta}$ for bosons, $\omega_n = \dfrac{\pi(2n+1)}{\hbar\beta}$ for fermions

These are well-known as the even and odd Matsubara frequencies.

Using this expansion we have

$$Z_0 = \int \left(\prod_{\vec{n}, n, \alpha} \frac{d\phi^*_{\vec{n},n,\alpha}\, d\phi_{\vec{n},n,\alpha}}{(2\pi i)^{(1\pm1)/2}} \frac{1}{(\hbar\beta)^{\pm}} \right)$$

$$\times \exp\left\{ \frac{-1}{\hbar} \sum_{\vec{n},n,\alpha} \phi^*_{\vec{n},n,\alpha} (-i\hbar\omega_n + \epsilon_{\vec{n},\alpha} - \mu) \phi_{\vec{n},n,\alpha} \right\}$$

* The difference of Jacobians in the bosonic and fermionic case, is a consequence of the fact that Grassmann variables we have that.

$$\int d\phi\, f_2 \phi = f_2 = \int d(f_2\phi) f_2 (f_2\phi).$$

* These are all Gaussian integral.

$$\Rightarrow Z_0 = \prod_{\vec{n},n\alpha} \left(\beta(-i\hbar\omega_n + \epsilon_{\vec{n},\alpha} - \mu)\right)^{\mp 1}$$

$$= \exp\left\{ \mp \sum_{\vec{n},n,\alpha} \ln\left(\beta(-i\hbar\omega_n + \epsilon_{\vec{n},\alpha} - \mu)\right) \right\}.$$

* To evaluate the sum over Matsubara frequencies, we need to add a convergenze factor $e^{i\omega_n\eta}$ and take limit $\eta \to 0$.

$$\Rightarrow \lim_{\eta\to 0} \sum_n \ln\left(\beta(-i\hbar\omega_n + \epsilon - \mu)\right) e^{i\omega_n\eta} = \ln\left(1 \mp e^{-\beta(\epsilon-\mu)}\right)$$

To check: differentiate it w.r.t. $\beta\mu$.

$$\Rightarrow \lim_{\eta\to 0} \frac{1}{\hbar\beta} \sum_n \frac{e^{i\omega_n\eta}}{i\omega_n - (\epsilon-\mu)/\hbar} = \mp \frac{1}{e^{\beta(\epsilon-\mu)} \mp 1}$$

for boson $\Rightarrow \frac{1}{e^{\beta(\epsilon-\mu)}+1}$

Need to prove!

<pf>

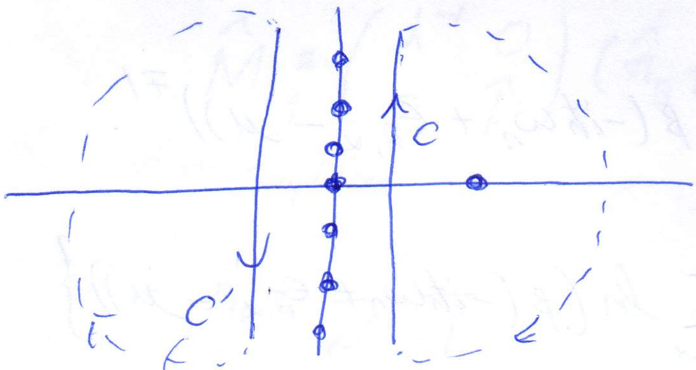
$\dfrac{\hbar\beta}{e^{\hbar\beta z}\mp 1}$ has poles at the $\underset{\text{boson}}{\underline{\text{even}}}$ and $\underset{\text{fermion}}{\underline{\text{odd}}}$.

Matsubara frequencies. with residue ± 1.

$\therefore$ By Cauchy's theorem, the l.h.s.

$$= \lim_{\eta \to 0} \frac{1}{2\pi i} \int_C dz \, \frac{e^{\eta z}}{z - (\epsilon - \mu)/\hbar} \, \frac{\pm 1}{e^{\hbar\beta z}} \Longrightarrow \mp \frac{1}{e^{\beta(\epsilon-\mu)} \mp 1}$$

$C :=$ contour



The integration of contour C' vanishes, since the integrant behaves as $\mp e^{-(\hbar\beta - \eta)\mathrm{Re}(z)}/|z|$. for $\mathrm{Re}(z) \to \infty$ and as $-e^{\eta \mathrm{Re}(z)}/|z|$ for $\mathrm{Re}(z) \to -\infty$.

$\therefore$ The integrant always $\to 0$. much faster than $1/|z|$. on C' for any $0 < \eta < \hbar\beta$.

Adding to the contour C with $\underset{0}{\overset{}{C'}}$, we can derive the result by Cauchy's Thm.

Prove: For fermionic fields the periodic boundary condition

$$\varphi(\vec{x},0)=\varphi(\vec{x},\beta).$$

is replaced by an antiperiodic boundary condition

$\psi(\vec{x},0)=-\psi(\vec{x},\beta)$ in order to reproduce the results of Chapter II.5.

Consider a simple formionic oscillator defined by the Lagrangian

$$L=ia^{+}(t)\dot{a}(t)-\omega a^{+}(t)a(t). \qquad (1)$$

The system has two energy levels ϵ_0 and ϵ_1 corresponding to the single fermion state being empty or occupied, and

$$\epsilon_0=E_0-\frac{1}{2}\omega,\quad \epsilon_1=E_0+\frac{1}{2}\omega. \qquad (2)$$

E_0 is arbitrary. It's clear that

$$\underbrace{tr\, e^{-iHT}}_{\shortparallel}=e^{-iE_0T}\left(e^{i\omega T/2}+e^{-i\omega T/2}\right). \qquad (3)$$

$$\int[da][da^{+}]\exp\left(i\int_0^T L\,dt\right)$$

$$\shortparallel$$

$$2e^{-iE_0T}\frac{Det[i\,d/dt-\omega]}{Det\left[i\frac{d}{dt}\right]}=2e^{-iE_0T}\prod_n\frac{\epsilon_n(\omega)}{\epsilon_n(0)}. \qquad (4)$$

Normalizasion: $tr\,e^{-iHT}\big|_{\omega=0}\equiv 2e^{-iE_0T}$ (5)

eigenvalues are defined by

$$i\frac{d}{dt}f_n(t)-\omega f_n(t)=\epsilon_n(\omega)f_n(t). \qquad (6)$$

$$f_n(t+T)=\pm f_n(t).$$

For periodic conditions,

$$\epsilon_n = \frac{-2n\pi}{T} - \omega, \quad n = 0, \pm 1, \pm 2, \cdots \qquad (7)$$

for antiperiodic

$$\rightarrow \epsilon_n = -\frac{(2n+1)\pi}{T} - \omega, \quad n = 0, \pm 1, \pm 2, \cdots \qquad (8)$$

Goal: To verify the eigenvalues in (4) will reproduce (3)

if choose antiperiodic boundary conditions.

for which $$\prod_{n=-\infty}^{\infty} \frac{\epsilon_n(\omega)}{\epsilon_n(0)} = \prod_{n=-\infty}^{\infty} \left(1 + \frac{\omega T}{(2n+1)\pi}\right)$$

$$= \cos\left(\frac{\omega T}{2}\right) \qquad (9)$$

Any free field theory of formions can be reduced to a sum of Lagrangian of the form (1).

* ψ stands for a set of N two-component fermion fields

$$\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(N)} \Rightarrow I_N(\sigma) = \int \prod_{j=1}^{N} [d\phi^{(j)}][d\bar{\psi}^{(j)}] \times \exp\left(i \sum_{k=1}^{N} \bar{\psi}^{(k)} (i\not{\partial} - \sigma)\psi^{(k)}\right)$$

$\llcorner (10)$

which defines $I_N(\sigma)$

It's immediate that $I_N(\sigma) = I_1[(\sigma)]^N$. $\qquad (11)$

Integrating over fermion fields I_1

$$\Rightarrow I_1(\sigma) = I_1(0) \frac{\operatorname{Det}[\gamma^0(i\not{\partial} - \sigma)]}{\operatorname{Det}[\gamma^0 i\not{\partial}]} = I_1(0) \prod \frac{\epsilon(\sigma)}{\epsilon(0)}$$

$\llcorner (12)$

where $\gamma^0(i\not{\partial} - \sigma)\xi = \epsilon\xi$

$\llcorner$ define this eigenvalue.

And $\underline{\xi(x, t+T) = -\xi(x, t)}$ *# $\rightarrow$ change notation, and let $t=0$, $T=\beta$

$\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta)$

\# we also have $\xi(x+L, t) = \xi(x, t)$.

Discuss the solitons in the so-called sine-Gordon theory

3 $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - g\cos(\beta\varphi)$.

(i) Find the topological current.

(ii) Is the $Q=2$ soliton stable or not?

(i) $*$ $\because$ Sine-Gordon theory has an infinite number of vacua occuring at $\varphi = (2n+1)\frac{\pi}{\beta}$.

$\therefore$ $\exists$ a whole spectrum of solitons, such that

there exists

$$\varphi(\pm\infty) = (2n_\pm + 1)\frac{\pi}{\beta}.$$

The topological current is

$$J^\mu = \left(\frac{\beta}{2\pi}\right)\varepsilon^{\mu\nu}\partial_\nu\varphi$$

with the corresponding charge $Q = (n_+ - n_-)$.

(ii) For $Q=2 \Rightarrow$ Soliton decays to two $Q=1$ solitons.

5. Show: within a region in which φ^a is constant, $F_{\mu\nu}$ as defined in the text is the electromagnetic field strength. Compute $\vec{B}$ far from the center of a magnetic monopole and show that Dirac quantization holds.

3

$\Rightarrow \varphi^a = v\delta^{a3} = \text{const.}$

Using $(D_\mu\varphi)^b = \partial_\mu \varphi^b + e\,\varepsilon^{bcd} A^c_\mu \varphi^d$

$$\left.\begin{aligned} \Rightarrow (D_\mu\varphi)^1 &= ev A^2_\mu \\ (D_\mu\varphi)^2 &= -ev A^1_\mu \end{aligned}\right\} \Rightarrow F_{\mu\nu} \equiv \frac{F^a_{\mu\nu}\varphi^a}{|\varphi|} - \frac{\left(\frac{1}{e}\right)\varepsilon^{abc}\varphi^a (D_\mu\varphi)^b (D_\nu\varphi)^c}{|\varphi|^3}$$

$$\Rightarrow \underset{\substack{\uparrow \\ A^3 \text{ is massless component of the Yang-Mills field.}}}{\partial_\mu A^3 - \partial_\nu A^3} = F^3_{\mu\nu} + e\left(A^2_\mu A^1_\nu - A^2_\nu A^1_\mu\right).$$

Let's compute $B_k = \varepsilon_{ijk} F_{ij}$ far away with magnetic monopole.

Focus on the term of order $1/r^2$ in $\vec{B}$.

$\because D_\mu\varphi \to O\left(\frac{1}{r^2}\right)$ by construction we can drop the second term in F_{ij}.

$\therefore$ Only have to compute $F^a_{ij} \equiv \partial_i A^a_j - \partial_j A^a_i + e\,\varepsilon^{abc} A^b_i A^c_j$

$\because F^a_{ij}$ will be contracted with $\frac{\varphi^a}{|\varphi|} = \frac{x^a}{r}$ in the end, so just drop some of the terms in F^a_{ij} for simple.

$$\Rightarrow \partial_i A^a_j = \partial_i\left(\frac{1}{e}\varepsilon^{ajl}\frac{x^l}{r^2}\right) \simeq \frac{1}{e}\varepsilon^{aji}\frac{1}{r^2}$$

$$e\,\varepsilon^{abc} A^b_i A^c_j = \frac{(1/e)\,\varepsilon^{abc}\varepsilon^{bim}\varepsilon^{cjn} x^m x^n}{r^4} = \Big\downarrow$$

$$\left(\frac{1}{e}\right)\left(\delta^{ci}\delta^{am} - \delta^{cm}\delta^{ai}\right)\varepsilon^{cjn}\hat{x}^m x^n / r^4 = \frac{1}{er^4}\varepsilon^{ijn}x^a x^n$$

$$\Rightarrow \frac{F_{ij}^a \varphi^a}{|\varphi|} = \frac{F_{ij}^a x^a}{r} = \frac{1}{er^3}(-2+1)\varepsilon^{aij}x^a = \frac{-1}{er^3}\varepsilon^{aij}x^a$$

hence $B_k = \frac{-1}{er^2}\hat{x}^k$.

Magnetic charge $\Rightarrow g = \frac{-4\pi}{e}$.

Introduce a Ψ field (which could be a Bose or Fermi field), transforming in the $I = \frac{1}{2}$ representation with the corresponding covariant derivative $D_\mu \Psi = \partial_\mu \Psi - ie\left(\frac{1}{2}\tau^a\right)A_\mu^a \Psi$.

The field Ψ carries electric charge $\frac{1}{2}e$.

$\therefore$ The fundamental unit-electric charge is $\frac{1}{2}e$ (not e

and the above result is change into $g = \frac{-4\pi}{e} = \frac{-2\pi}{\left(\frac{e}{2}\right)}$

which is the same as Dirac quantization condition.

.12. Evaluate $n \equiv -\frac{1}{24\pi^2}\int_{S^3} tr\,(g\,dg^{\dagger})^3$ for the map $g = e^{i\vec{\theta}\cdot\vec{\sigma}}$.

3

$$g = e^{i\vec{\theta}\cdot\vec{\sigma}} \underset{\substack{\uparrow \\ \text{near the} \\ \text{identity element.}}}{\simeq} 1 + i\vec{\theta}\cdot\vec{\sigma} \Rightarrow g\,dg^{\dagger} \simeq -i\,d\vec{\theta}\cdot\vec{\sigma}.$$

In a small neighborhood of the identity element the group manifold is locally Euclidean

$$\therefore\ tr\,(g\,dg^{\dagger})^3 = i\,tr\,(\sigma^i\sigma^j\sigma^k)\,d\theta^i d\theta^j d\theta^k$$
$$= -12\,d\theta^1 d\theta^2 d\theta^3.$$

is manifestly proportional to the volume element on S^3.

For $g = e^{i(\theta_1\sigma_1 + \theta_2\sigma_2 + m\theta_3\sigma_3)}$.

$$tr\,(g\,dg^{\dagger})^3 = -12\,m\,d\theta^1 d\theta^2 d\theta^3.$$

1.13. Prove: Higher order corrections do not change the chiral anomaly $\partial_\mu J_5^\mu = \left[\frac{1}{(4\pi)^2}\right] \varepsilon^{\mu\nu\lambda\sigma} \text{tr} F_{\mu\nu} F_{\lambda\sigma}$.

1

Start with integrat $\partial_\mu J_5^\mu$.

$$\Rightarrow \int d^4x \left(\partial_\mu J_5^\mu\right) = \int d^3x\, J_5^0 \Big|_{t=+\infty} - \int d^3x\, J_5^0 \Big|_{t=-\infty}$$

Recalling $J_5^0 = \psi_R^\dagger \psi_R - \psi_L^\dagger \psi_L$.

thus two spatial integrations

$= \begin{pmatrix} \text{\# of right moving} \\ \text{fermion quanta} \end{pmatrix} - \begin{pmatrix} \text{\# of left moving} \\ \text{fermion quanta} \end{pmatrix}$.

at $t = \pm\infty$.

$\therefore \int d^4x \left(\partial_\mu J_5^\mu\right)$ is an integer.

On the other hand, in the textbook, the author have been proved that $\int \text{tr} F^2$ is a topological invariant. i.e. with suitable normalization,

$\frac{1}{(4\pi)^2} \int d^4x\, \varepsilon^{\mu\nu\lambda\sigma} \text{tr} F_{\mu\nu} F_{\lambda\sigma}$ is an integer.

$\therefore \frac{1}{(4\pi)^2}$ cannot be shifted

"even" a little bit by quantum fluctuation.