Miscellaneous (Beyond the scope of your exams)

1. How to complete the square in a given quadratic function?

2. How to find maximum or minimum using the vertex form?

3. Find the tangent line given a point on a parabola only using the notions we learned in this class so far (the definition of linear function, and its slope, the quadratic formula, etc).

4. How to drew the parabola either with or without giving the quadratic function, and only using a ruler and a compass/string?
   => The standard form.

5. Why the point P introduced in 4 is called the focus or focal point, and r in the standard form is called the focal length?

Solving $f(x)=0$ for x (The method of completing the square)

$f(x) = ax^2+bx+c$, $a \neq 0$.

Note: This is an alternative method to derive the vertex form. cf. The method introduced on P37 is starting from the base function $g(x) = x^2$; here, we start from the general form.

Let $f(x)=0$

$$\Rightarrow a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

$$\Rightarrow a\left(x^2 + 2 \cdot \frac{b}{2a}x\right) + c = 0$$

$$\Rightarrow a\left(x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2\right) + c - \left(\frac{b}{2a}\right)^2 a = 0$$

$$\Rightarrow a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a} = 0.$$

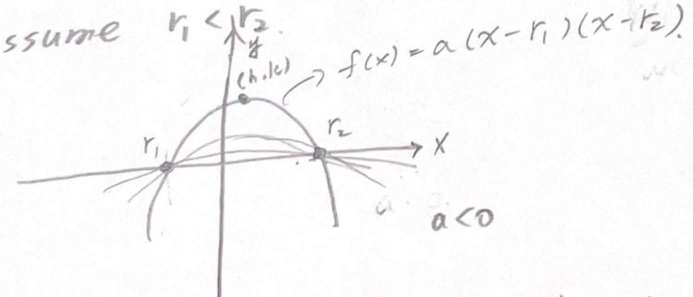
$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}.$$

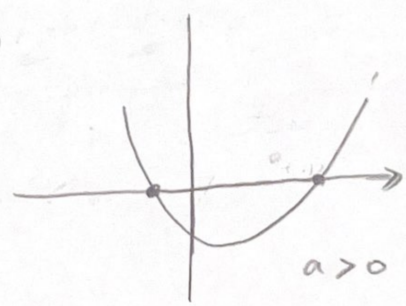
$$\Rightarrow \left(x + \frac{b}{2a}\right) = \frac{\pm\sqrt{b^2-4ac}}{2a}.$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = r_1 \text{ or } r_2 \text{ where } \begin{cases} r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \\ r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} \end{cases}$$

Geometrically, it has the following visualization:

Assume $r_1 < r_2$.





$\Rightarrow (x-r_1)(x-r_2)=0$ with $a \neq 0$.

if a is fixed, then the graph is unique.

Let $f(x) = a(x-h)^2+k$, $a\neq 0$, $x\in(-\infty,\infty)$. Then there are

two cases

(i) $a>0$

(ii) $a<0$.

(i) $a>0$. Since $(x-h)^2$ is alway non-negative and increasing as $x\to\pm\infty$, if we let $x=h$, then $f(h)=k$ is the minimum.

(ii) $a<0$. Likewise, $(x-h)^2\geq 0$, but $a<0$, so $a(x-h)^2\leq 0$, for all x in $(-\infty,\infty)$. Then, if $x=h$, we have the maximum value of the function $f(x)$.

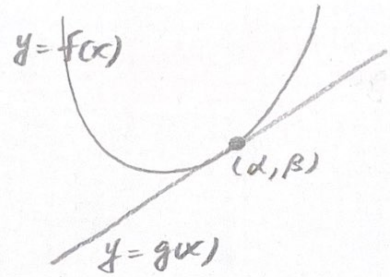
Given a quadratic function, and a point on the graph of the function. The tangent line at that point of the graph can be derived by Using the quadratic formula, and the function that describes the change of the slope of the tangent function is linear.

$\Rightarrow$ This is an application of the notion of composition function, and quadratic formula.

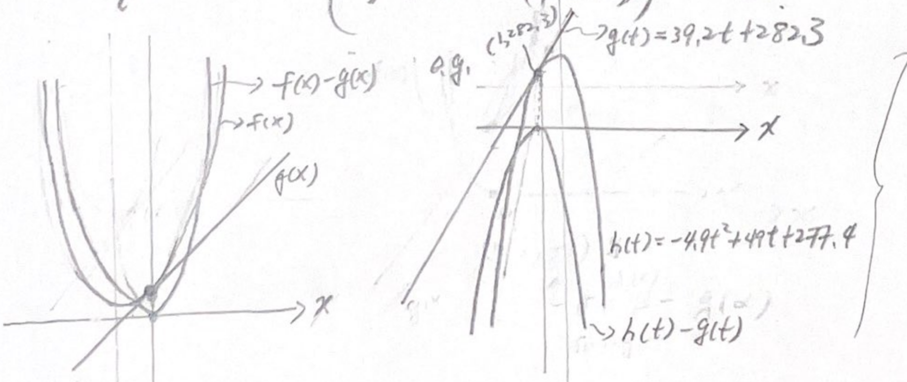
<u>Example</u>

Let $f(x) = ax^2 + bx + c$, $a \neq 0$, and (α, β) be a point such that $f(\alpha) = \beta$.

Let the tangent line of the graph $y = f(x)$ at (α, β) be $g(x) = mx + k$



To find m and k, we solve the quadratic equation $(f(x) - g(x)) = 0$



geometric figures

$\Rightarrow f(x) - g(x) = ax^2 + (b-m)x + (c-k) = 0$

Since we have $(b-m)^2 - 4a(c-k) = 0$, ($\therefore r_1 = r_2$, double root)

the quadratic formula gives $\alpha = \dfrac{-(b-m)+0}{2a} \Rightarrow m = b + 2a\alpha$

Since $g(\alpha) = \beta$

Then $\beta = m\alpha + k \Rightarrow k = \beta - m\alpha = \beta - (b+2a\alpha)\alpha$

It follows that $g(x) = (b+2a\alpha)x + (\beta - (b+2a\alpha)\alpha)$

Furthermore, if we focus on the rate of change on the slope of $y = g(x)$ and arbitrarily choose two data points, e.g. $x_1 = 0$, and $x_2 = 1$.

Then we can observe that if $x_1 \neq x_2$ then the value (it's actually a slope)

$$\frac{(b+2ax_2) - (b+2ax_1)}{x_2 - x_1}$$

$= 2a$ which is a constant.

Hence, we can write down the function that describes the change of $g(x)$:

$$V(x) = 2ax + V_0 \quad \text{where } V_0 = b + 2a\cdot 0 \quad \because x = 0.$$

$$= 2ax + b$$

<u>Example</u> Use the above result, we can find $V(t)$ in Example 1.

<u>Sol</u>. $V(t) = 2\cdot(-4.9)t + 49$

$= -9.8t + 49$. (m/s)

Standard form

Let E be a 2d plane. On E, there are a line L, and a point P, where P is not on L. Then, the trajectory of a moving point Q is a parabola, if the distance between Q and L, and the distance between Q and P are always identical. Note: L is called the diretrix and P is the focal point of the parabola.

Example.

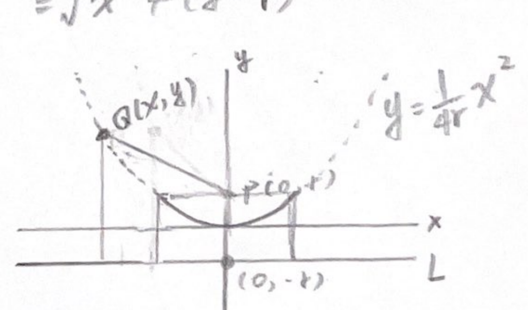
Let $L: y = -r$, $r > 0$. P is a point on y-axis, $P = (0, r)$. Let $Q = (x, y)$

The distance between Q and L.

$$d(Q,L) = \frac{|ax+by-k|}{\sqrt{a^2+b^2}} = \frac{|y+r|}{\sqrt{1^2+0^2}} = |y+r|$$

↑ if L is $ax+by-k=0$. $Q=(x,y)$.

$$d(P,Q) = \sqrt{(x-0)^2+(y-r)^2} = \sqrt{x^2+(y-r)^2}$$



$$\Rightarrow |y+r| = \sqrt{x^2+(y-r)^2}$$

$$\Rightarrow (y+r)^2 = x^2+(y-r)^2$$

$$\Rightarrow 2yr = x^2 - 2yr$$

$$\Rightarrow \boxed{y = \frac{1}{4r}x^2}$$ — the standard form.

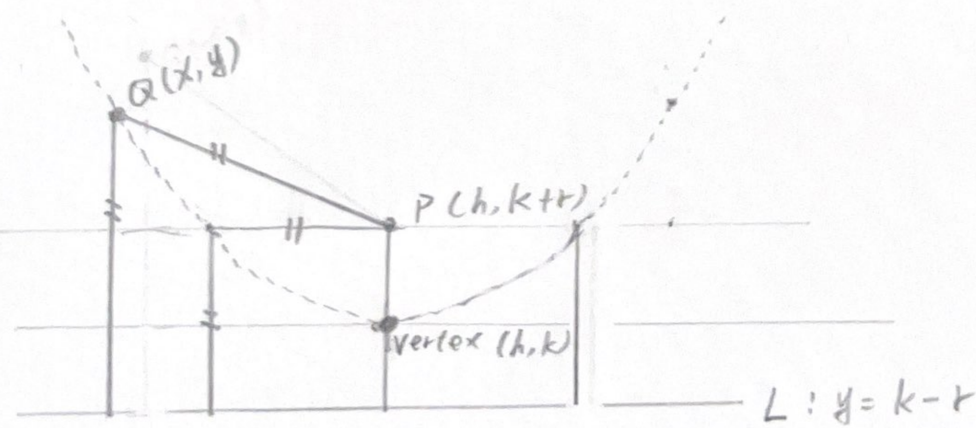
By comparing the coefficients between $\underbrace{\text{the standard form}}_{y=\frac{1}{4r}x^2}$ and $\underbrace{\text{the base function}}_{y=g(x)=x^2=1\cdot x^2}$ written in the general form, then we can see that the focus length r in the case of the base function is $\frac{1}{4}$.

In other words, the diretrix of $g(x)$ is $y=\frac{-1}{4}$.

Also, the derivation also shows the geometric meaning of the coefficient "a" in the vertex form.

$$\Rightarrow y = a(x-h)^2+k = \frac{1}{4r}(x-h)^2+k$$

To manually plot this parabola (using a rular and a string)



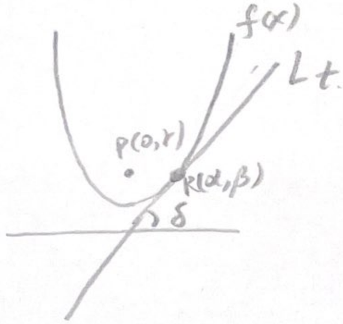
Let $f(x) = \frac{1}{4r}x^2$, $P:(0,r)$, $Q:(x,y)$, and $R:(\alpha,\beta)$

where R is a point on $y = f(x)$.

Then we can show $\tan\delta = \frac{\pi}{2}$, where δ is the angle

(After you learn trigonometric functions, you can come back to read this.)

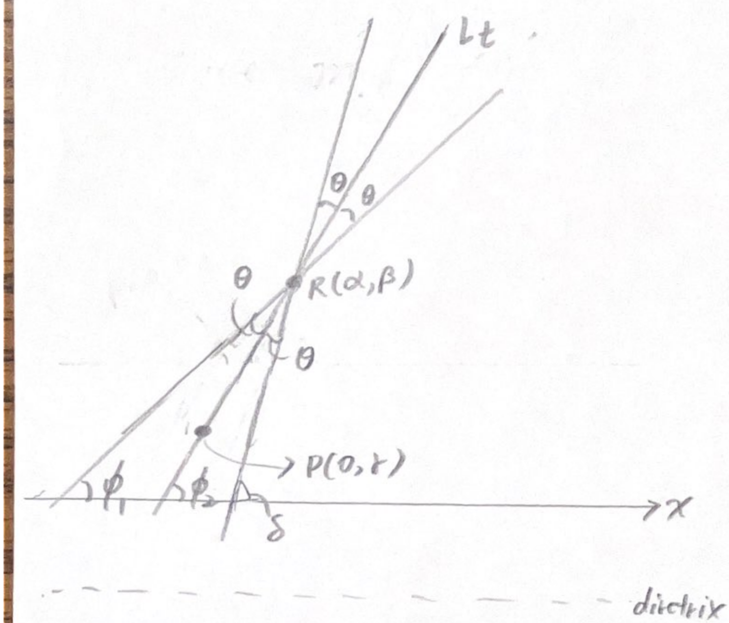
between the tangent line $\hat{L_t}$ and the x axis.



Since $f(x) = \frac{1}{4r}x^2$, we have $L_t : g(x) = \frac{\alpha}{2r}x + \left(\beta - \frac{\alpha^2}{2r}\right)$

$\Rightarrow b = 0,\ a = 1/4r,\ c = 0$

and $\overline{PR} : y = \left(\frac{\beta - r}{\alpha}\right)x + \left(r - \frac{\beta - r}{\alpha}\right)$.



Then, $m_{PR} = \tan\phi_2 = \frac{\alpha}{2r}$,

$m_{L_t} = \tan\phi_1 = \frac{\beta - r}{\alpha}$.

Additionally, $\delta = \phi_1 + 2\theta$, and $\phi_2 = \theta + \phi_1$.

Hence, $\delta = 2\phi_2 - \phi_1$.

$$\tan 2\phi_2 = \frac{2\tan\phi_2}{1 - \tan^2\phi_2} = \frac{4r\alpha}{4r^2 - \alpha^2}$$

$$\tan\delta = \tan(2\phi_2 - \phi_1) = \frac{\tan 2\phi_2 - \tan\phi_1}{1 + \tan 2\phi_2 \tan\phi_1}$$

$\to$ Since $\tan 2\phi_2 \tan\phi_1 = \frac{4r\alpha}{4r^2 - \alpha^2}\left(\frac{\beta - r}{\alpha}\right) = \frac{4r\alpha}{4r^2 - \alpha^2}\cdot\left(\frac{\alpha^2}{4r} - r\right) = -1$

$\Rightarrow \delta \overset{\to\infty}{=} \pi/2$.