Conway's Basic Theorem on Rational Tangles

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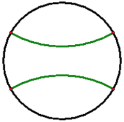
Overview

Definition.

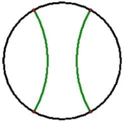
Tangle diagram

which is a diagram contained in a disk that consists of two strands whose four endpoints are fixed along the boundary of the disk.

A tangle diagram can be viewed as a subdiagram of a knot or link by thinking of the two strands in the tangle extend into a larger knot or link diagram outside of the disk. If we begin at one of the free ends emanating from the disk and walk along it, we will enter the disk, and eventually leave the disk to meet another end of strand.



$[0]$



$[\infty]$

Definition.

Two tangle diagrams are equivalent

which provided that one can be continuously deformed into the other via planar isotopies and Reidemeister moves performed within the disk while keeping the four endpoints of the two strands fixed.

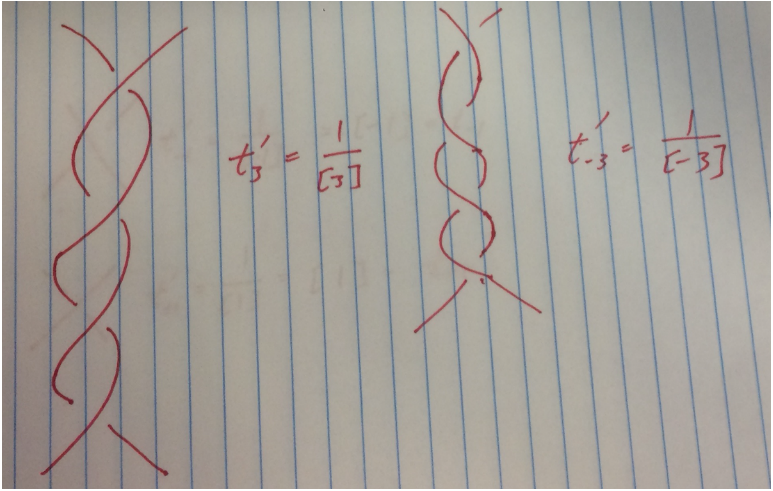
The sim symbol: $\sim$ denotes tangle equivalence.

Definition.

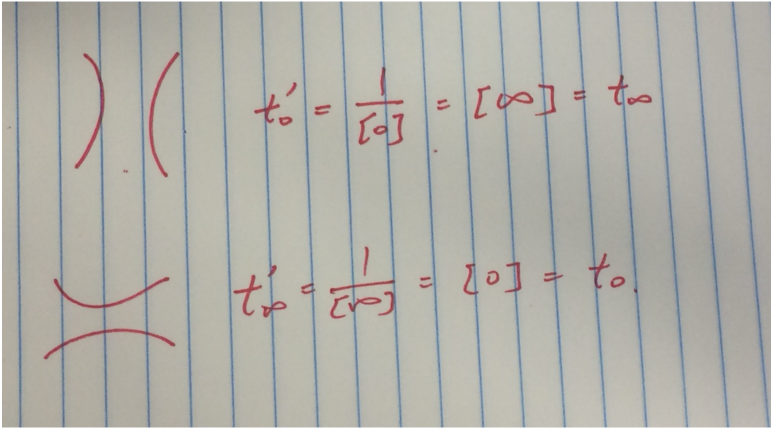
Horizontal integer tangle t_a (Vertical integer tangle t'_a)

a twist of two strands $|a|$ times in the positive or negative according to the signs of a.

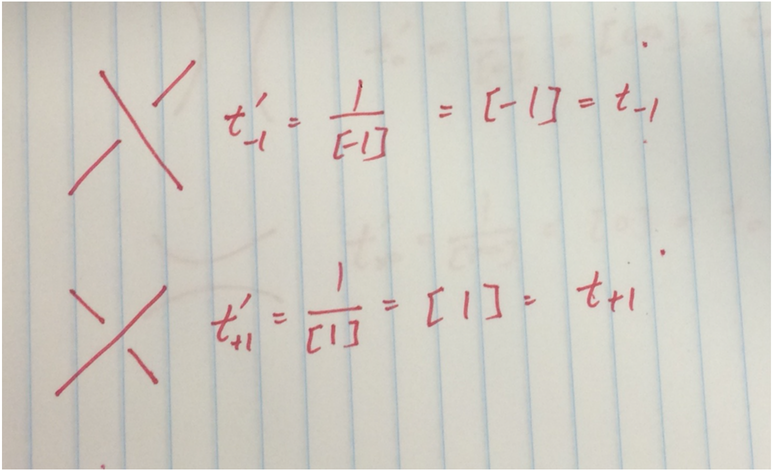
Examples of tangles



Examples of tangles



Examples of tangles



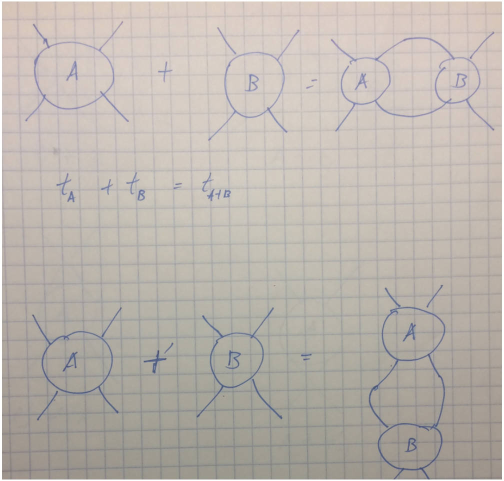
Definition. 2 basic operations of basic tangles.

Definition. A tangle is in standard form which is called a basic tangle if it is created by consecutive additions, $(+)$, of simple tangles from the right and multiplications, $*$ or $(+')$, by simple tangles from the bottom.

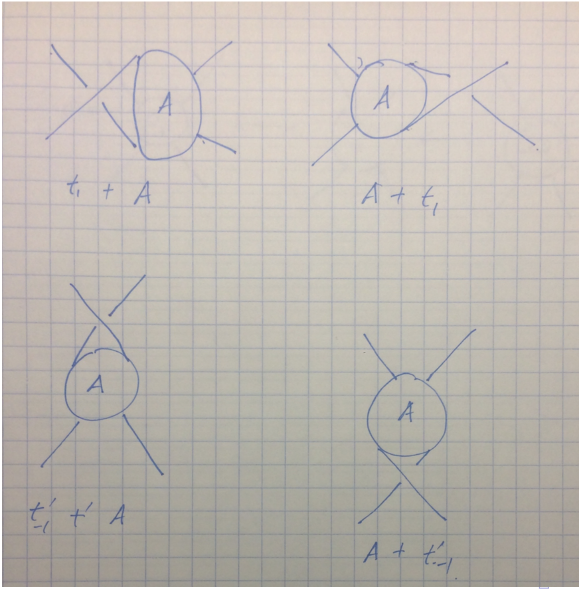
Basic Tangles in a recursive fashion

- basic horizontal tangles: $(((t_a +' t'_b) + t_c) +' t'_d) + \ldots$
- basic vertical tangles: $(((t_a + t'_b) +' t_c) + t'_d) +' \ldots$

Basic operations

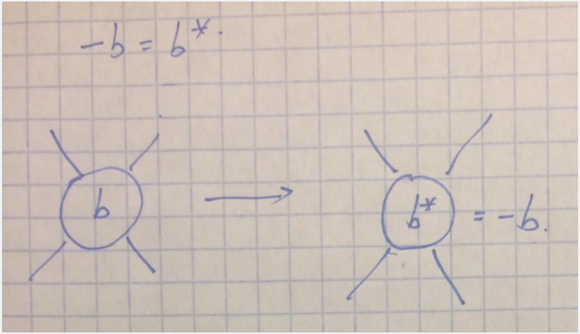


Basic operations



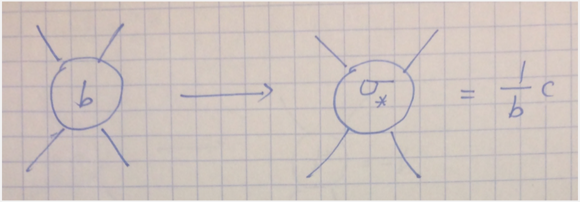
3 operations to classify the basic tangles (1)

$-b = b*$ is the mirror image of b, reversing all crossings



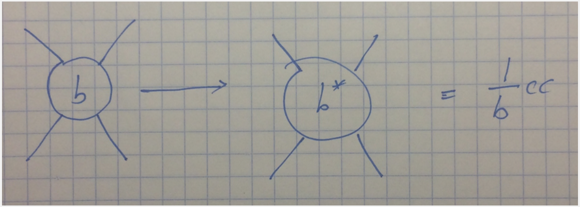
3 operations to classify the basic tangles (2)

$c - inverse, \frac{1}{b}c,$ rotating b by 90 degree clockwise and taking the mirror image



3 operations to classify the basic tangles ()

$cc - inverse$, $\frac{1}{b}cc$, rotating b by 90 degree counterclockwise and taking the mirror image



6 results derived from the 5 operations and the definition of basic tangles.

1. b is a basic horizontal tangle $\Rightarrow \frac{1}{b}c$ is a basic vertical tangle.
2. b is a basic vertical tangle $\Rightarrow \frac{1}{b}cc$ is a basic horizontal tangle.
3. $t'_a = \frac{1}{t_a}c = \frac{1}{t_a}cc$ and $t_a = \frac{1}{t'_a}c = \frac{1}{t'_a}cc$
4. $t_d + t'_e = t_d + \frac{1}{t_e}$
5. $$T +' t'_n = \frac{1}{t_n + \frac{1}{T}c}cc \Leftrightarrow T * \frac{1}{[n]} = \frac{1}{[n] + \frac{1}{T}}$$

   any rational tangle can be built by a series of the following operations: Addition of [1] and Inversion.
6. $-(b+c) = (-b) + (-c), -(\frac{1}{b}) = \frac{1}{-b}, -t_a = t_{-a}$, and $-t'_a = t'_{-a}$.

Facts of basic tangles.

Example (by directly using 5 we can go into a form of continued fraction from basic tangles)

$$t_3 \to t_3 +' t_2$$

$$\sim t_3 \to \frac{1}{t_2 + \frac{1}{t_3}c}cc$$

New definitions of basic tangles.

Example

$$a_n + \frac{1}{a_{n-1} + \frac{1}{\ldots + \frac{1}{a_1}}} \sim t_{a_n} + \frac{1}{t_{a_{n-1}} + \frac{1}{\ldots + \frac{1}{t_{a_1}}}}$$

any basic horizontal tangle b can be written in the form on the right hand side.

Example

$$\frac{1}{a_n + \frac{1}{a_{n-1} + \frac{1}{\ldots + \frac{1}{a_1}}}} \sim \frac{1}{t_{a_n} + \frac{1}{t_{a_{n-1}} + \frac{1}{\ldots + \frac{1}{t_{a_1}}}}}$$

any basic vertical tangle b can be written in the form on the right hand side.

Definition.

Fraction of tangles.

Denote it by $F(t_a) = a \in \mathbb{Q}$, and $F(t'_a) = F(\frac{1}{t_a}) = \frac{1}{a} = \frac{1}{F(t_a)} \in \mathbb{Q}$.

If b is a basic tangle, then by definitions, and the facts of basic tangles:

1. $F(\frac{1}{b}) = \frac{1}{b}$, independent of c or cc.
2. $t_a + b$ is basic
   $\Rightarrow F(t_a + b) = F(t_a) + F(b) = a + F(b)$
3. $F(-t_a) = F(t_{-a}) = -a = -F(t_a)$
4. $F(-t'_a) = F(t'_{-a}) = F(\frac{1}{t_{-a}}) = \frac{-1}{a}$
   $= -F(\frac{1}{t_a}) = -F(t'_a)$
5. $F(-b) = -F(b)$

$\Rightarrow$

Theorem (Conway's Basic Theorem)

Let T_1 and T_2 be the given basic tangles.
$F(T_1) = F(T_2) \Rightarrow T_1$ *is ambient isotopic to* T_2.

Recall:

$$\frac{p}{q} = a_n + \frac{1}{a_{n-1} + \frac{1}{\ldots + \frac{1}{a_1}}} \text{ (which is a regular continued fraction)}$$

$$= F\left(t_{a_n} + \frac{1}{t_{a_{n-1}} + \frac{1}{\ldots + \frac{1}{t_{a_1}}}}\right) := F(T) \in \mathbb{Q}$$

$a_i > 0,\ i = \{1, 2, \ldots, n-1\}$.

To convert $F(T)$ into the regular continued fraction.

The key is Lagrange's formula.

$$a - \frac{1}{b} = (a-1) + \frac{1}{1 + \frac{1}{(b-1)}}$$

By repeatedly using this formula, a negative sign can only possibly exist in front of the first term.

Example.

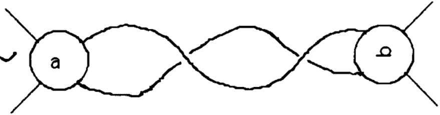
$$\frac{43}{62} = 1 + \frac{1}{-3 + \frac{1}{-4 + \frac{1}{5}}} = 1 - \frac{1}{3 + \frac{1}{4 - \frac{1}{5}}}$$

$$= 1 - \frac{1}{3 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}}} = 0 + \frac{1}{3 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4}}}} = \frac{1}{3 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4}}}}$$

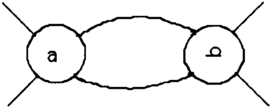
$$= \textit{a continued fraction with all terms positive}$$

Topological version of Lagrange's formula.

$$[a] + [-1] + \cfrac{1}{[1] + \cfrac{1}{[b-1]}} \sim$$



$$\sim$$



$$[a] + \frac{1}{[-b]} \text{cc}$$

Conversely...

Theorem (Conway's Basic Theorem)

Let T_1 and T_2 be the given basic tangles. If $T_1 \sim T_2$, then $F(T_1) = F(T_2)$.

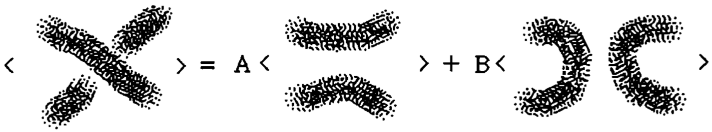
Strategy: Define an invariant, say $C(T)$, for arbitrary tangles T. And then try to make this satisfy the definition of regular fraction that

1. $C(T) = F(T)$, if T is rational basic tangles.
2. if $T \sim S$, then $C(T) = C(S)$

Let's develop this invariant $C(T)$ based on the bracket model of the Jones polynomials.

Developing the invariant, $C(T)$

(1)

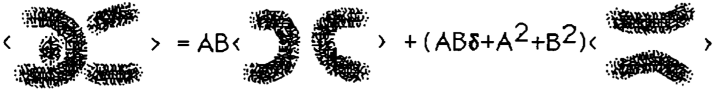


(2) $\langle OK \rangle = \delta \langle K \rangle,$

$\langle O \rangle = 1,$

Developing the invariant, $C(T)$

To build the invariant, we specialize this polynomial **for any tangles** by taking $\delta = -A^2 - A^{-2}$ and $B = A^{-1}$. Hence for the second Reidemeister move we have



Developing the invariant, $C(T)$

Definition.

Let T be a given tangle. Then we have

$$< T >= \alpha(T) < || > +\beta(T) <=>$$

$$= \alpha(T) < [\infty] > +\beta(T) < [0] >,$$

where $\alpha(T)$ and $\beta(T)$ are derived from (1) and (2) repeatedly until only the $[\infty]$ and $[0]$ basic tangles are left.

Theorem.

$\forall A, R_T(A) = \frac{\alpha(T)}{\beta(T)}$ is an ambient isotopic invariant of tangles. (Under the Reidemeister moves I, II and III.)

Developing the invariant, $C(T)$

We now specialize our bracket polynomial again: Let

$$A = \sqrt{i}, i^2 = -1 \Rightarrow B = \frac{1}{\sqrt{i}}, \delta = -A^2 - A^{-2} = 0.$$

Hence $< O >= 0$

Developing the invariant, $C(T)$

Definition.

$$C(T) = -iR_T(\sqrt{i})$$

Claim.

$$C(T) := -iR_T(\sqrt{i}) = F(T)$$

Need to show: $C\left(\frac{1}{T}\right) = \left(\frac{1}{C(T)}\right)^* = \frac{\alpha(T)}{\beta(T)}$

Developing the invariant, $C(T)$

By definition $< T >= \alpha(T) < [\infty] > +\beta(T) < [0] >$, $\Rightarrow < 1/T >= \alpha(T)^* < [0] > +\beta(T)^* < [\infty] >$ where $1/[0] = [\infty]$, and $(1/\sqrt{i})^* = \sqrt{i}$. It follows that

$$C\left(\frac{1}{T}\right) = \frac{\beta(T)^*}{i\alpha(T)^*} = \frac{i^*\beta(T)^*}{\alpha(T)^*} = \left(\frac{1}{C(T)}\right)^*.$$

The values of $C(T)$on rational tangles are rational numbers, thus

$$C\left(\frac{1}{T}\right) = \frac{1}{C(T)}$$

Developing the invariant, $C(T)$

Need to show: $C(T+S) = C(T) + C(S)$.

Proposition.

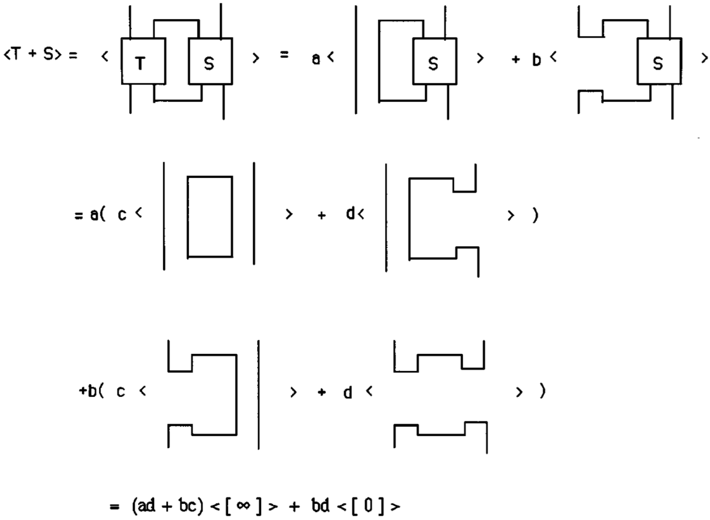
$$< T >= a < [\infty] > +b < [0] >$$
$$< S >= c < [\infty] > +d < [0] >$$

Then

$$< T + S >= (ad + bc) < [\infty] > +bd < [0] > .$$

Thus, $C(T+S) = C(T) + C(S)$ (since $\frac{ad+bc}{bd} = \frac{a}{b} + \frac{c}{d}$.)

Developing the invariant, $C(T)$



Reference

. R. Goldman, and L. H. Kauffman (1997)
Rational Tangles
ADVANCES IN APPLIED MATHEMATICS 18, 300 – 332.

Thank You!

Some fun stuffs after this slide, if we still have time.

Observation.

If we begin at one of the free ends emanating from the disk and walk along it, we will enter the disk, and eventually leave the disk to meet another end of strand.

#(ends)=even.

A disk must have an even number of ends.

Theorems.

Theorem (Flip Theorem)

A 180 degree rotation of a rational tangle b in the horizontal or vertical axis is ambient isotopic to b.

Theorem

The c-inverse and cc-inverse of a rational tangle are ambient isotopic.

Theorem

Every rational tangle is isotopic to a basic tangle.