19.1 The Axial Current in Two Dimensions.

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Goal: To study the physics that violates axial current conservation in a context in which the calculations are relatively simple. (2D massless QED)

$$\Rightarrow \mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}(F_{\mu\nu})^2. \quad \text{——— (19.2)}$$

$\mu, \nu \in \{0,1\}$.

$D_\mu = \partial_\mu + ieA_\mu$.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad \text{——— (19.3)}$$

In 2D, this set of relations can be represented by 2x2 matrices: $\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ —(19.4)

The Dirac spinor will be two-component field.

$$\gamma^0\gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{——— (19.5)}$$

As in four dimensions, there are two possible currents

$$\begin{cases} j^\mu = \bar{\psi}\gamma^\mu\psi \\ j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi. \end{cases} \quad \text{——— (19.6)}$$

and both are conserved if there is no mass term in the Lagrangian. To make the conservation law explicit, let's label the fermion field ψ in the spinor basis as $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$. ——— (19.7)

The subscript indicate the γ eigenvalue.

Using the explicit representations (19.4) & (19.7)

$\Rightarrow$ rewrite the fermion part of (19.2).

as $\mathcal{L} = \psi_+^\dagger i(D_0 + D_1)\psi_+ + \psi_-^\dagger i(D_0 - D_1)\psi_-$ — (19.8)

For a free theory, the field equation of ψ_+ would be $i(\partial_0 + \partial_1)\psi_+ = 0$. — (19.9)

The solution to this equation are waves that move to the right in the one dimensional space at the speed of light.

$\Rightarrow$ $\begin{cases} \psi_+ : \text{right-moving fermions} \\ \psi_- : \text{left-moving fermions.} \end{cases}$

$\rightarrow$ analogous to 4D.

Since the Lagrangian $\mathcal{L}$ (19.8) without mix left- and right-moving fields, it's obviously the # currents for these fields are separately conserved.

$$\therefore \partial_\mu \left(\bar{\psi} \gamma^\mu \left(\frac{1-\gamma^5}{2} \right) \psi \right) = 0 \qquad (19.10)$$

$$\partial_\mu \left(\bar{\psi} \gamma^\mu \left(\frac{1+\gamma^5}{2} \right) \psi \right) = 0.$$

In 1+1 D, the vector and axial vector fermionic currents are not independent of each other.

Let $\epsilon^{\mu\nu}$ = totally antisymmetric symbol in 2D.

$\epsilon^{01} = +1$, then $\gamma^\mu \gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$ — (19.11).

The currents $j^{\mu 5}$ and j^{μ} have the same relation.
$\therefore$ We can study the properties of the axial vector current by using results that just like for the vector current.

⓪ Vacuum Polarization Diagrams.

Recall the lowest-order vacuum polarization of QED in dimensional regularization. For the limit of zero mass, in eq. (7.90),

$$i\Pi^{\mu\nu}(q) = -i(q^2 g^{\mu\nu} - q^\mu q^\nu)\frac{2e^2}{(4\pi)^{d/2}}\, tr[1]\int_0^1 dx\, x(1-x)\frac{\Gamma(2-\frac{d}{2})}{(-x(1-x)q^2)^{2-\frac{d}{2}}} \qquad (19.12)$$

$tr[1] = 4 \xrightarrow{in} (7.88)$

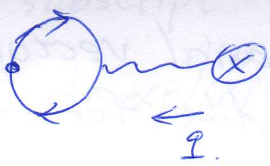
Now, set $tr[1]=2$ to be consistent with (19.4) and set $d=2$ in (19.12)

$$\Rightarrow i\Pi^{\mu\nu}(q) = i(q^2 g^{\mu\nu} - q^\mu q^\nu)\frac{2e^2}{4\pi}\cdot 2\cdot\frac{1}{q^2}$$

$$= i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{e^2}{\pi} \qquad (19.13)$$

Note also that $\underset{(pho.)}{m_\gamma^2} = \frac{e^2}{\pi}$ ——— (19.14)

Schwinger showed that this result is exact, and the photon of 2d-QED is a free massive boson.
As long as we obtain an explicit expression for the vacuum polarization, we can find the E.V. of the current by a background EM-field.
This quantity is generated by the diagram of the picture.



where $A_\nu(q)$ is the Fourier transform of the background field. which is satisfy the current conservation relation $q_\mu \langle j^\mu(q)\rangle = 0$.

And, $\langle j^{\mu 5}(q)\rangle = -\epsilon^{\mu\nu}\langle j_\nu(q)\rangle$

$$= \epsilon^{\mu\nu}\frac{e}{\pi}\left(A_\nu(q) - \frac{q_\nu q^\lambda}{q^2}A_\lambda(q)\right) \qquad (19.16)$$

If the axial vector current were conserved, this object would satisfy the Ward identity. Instead, we have $q_\mu\langle j^{\mu 5}(q)\rangle = \frac{e}{\pi}\epsilon^{\mu\nu} q_\mu A_\nu(q)$

$$(19.17)$$

which is the Fourier transform of the field equation $\partial_\mu j^{\mu 5} = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$ ——— (19.18)

$\hookrightarrow$ axial current is not conserved.

How? → The problem must come in the regularization of the vacuum polarization diagram. By dimensional analysis, we know this diagram has the form:

 $= ie^2\left(Ag^{\mu\nu} - B\frac{q^\mu q^\nu}{q^2}\right)$ ——— (19.19)

where B is a finite integral, A is also a integral in logarithmically divergent, so A is depends on the regularization. Dimensional regularization subtracts this integral to set $A=B$ then the vector current Ward identity is satisfied. But this leads to (19.17). So, instead to do this, one can regularize the integral A so that $A=0$.

Work out as previous step $\Rightarrow q_\mu \langle j^{\mu 5}(q)\rangle = 0$,

$$\text{but} \quad q_\mu \langle j^\mu(q)\rangle = \frac{e}{\pi} q^\nu A_\nu(q) \qquad (19.20)$$

However, the result in (19.20) is bad! Since it depends on the unphysical gauge d.o.f. of the vector potential. We conclude that it is not possible to regularize 2-d QED, such that the theory is gauge invariant and the axial vector current is conserved.

* The price of requiring gauge invariance is the anomalous nonconservation of the axial current shown in (19.18).

⊙ The Axial Vector Current Operator Equation.

Goal: To understand the axial current from another viewpoint by studying the operator equation for the divergence of $j^{\mu 5}$.

Varying the Lagrangian (19.2), derive the e.o.m. for fermion fields:

$$\not{\partial}\psi = -ie\not{A}\psi, \qquad \text{—— (19.21)}$$
$$\partial_\mu \bar{\psi}\gamma^\mu = ie\bar{\psi}\not{A}$$

Using (19.21) => $\partial_\mu j^{\mu 5} = 0$. But, a closer look at these manipulations reveals some subtleties, which alter the final conclusion.

(Def) $$j^{\mu 5} = \underset{\epsilon\to 0}{\text{symm}\lim}\left\{\bar{\psi}(x+\tfrac{\epsilon}{2})\gamma^\mu\gamma^5 \exp\left[-ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} dz\cdot A(z)\right]\cdot\psi(x-\tfrac{\epsilon}{2})\right\} \text{ —— (19.22)}$$

Notice that, ∵ ψ, $\bar{\psi}$ are at different points, ∴ we should introduce a Wilson line (15.53) in order that the operator be locally gauge invariant. To give $j^{\mu 5}$ the correct transformation properties under Lorentz transformations, the $\lim_{\epsilon\to 0}$ should take symmetrically,

$$\underset{\epsilon\to 0}{\text{symm}\lim}\left\{\frac{\epsilon^\mu}{\epsilon^2}\right\} = 0, \quad \underset{\epsilon\to 0}{\text{symm}\lim}\left\{\frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2}\right\} = \frac{1}{d}g^{\mu\nu} \quad (d=4)$$
(19.23)

$$\Rightarrow \overline{\bar{\psi}(x+\tfrac{\epsilon}{2})\Gamma\psi}(x-\tfrac{\epsilon}{2}) = \frac{-i}{2\pi}\, tr\left[\frac{\gamma^\alpha \epsilon_\alpha}{\epsilon^2}\Gamma\right] \quad \text{—— (19.27)}$$

Note: (19.27) contains an extra minus sign from the interchange of fermion operators.

$\because$ The contraction of fermion fields is singular $\epsilon \to 0$, the terms of order ϵ in the (19.25) can give a finite contribution.

By contracting the fields and (19.27),

$$\partial_\mu j^{\mu 5} = \underset{\epsilon\to 0}{\text{symm lim}} \left\{ \frac{-i}{2\pi}\, tr\left[\frac{\gamma^\alpha \epsilon_\alpha}{\epsilon^2}\gamma^\mu \gamma^5\right] \cdot (-ie\epsilon^\nu F_{\mu\nu}) \right\} \quad \text{—— (19.28)}$$

In 2d, $tr[\gamma^\alpha \gamma^\mu \gamma^5] = 2\epsilon^{\alpha\mu}$, $\therefore \partial_\mu j^{\mu 5}$

$$= \frac{e}{2\pi} \underset{\epsilon\to 0}{\text{symm lim}} \left\{ \frac{2\epsilon_\mu \epsilon^\nu}{\epsilon^2} \right\} \epsilon^{\mu\lambda} F_{\lambda\nu} \quad \text{—— (19.29)}$$

* If we define the axial vector current by reversing the sign of the Wilson line in (19.22), we would find the various contributi[ons] canceling on the r.h.s. of (19.29).

An example with "Fermion Number Nonconservation"

Goal: To show the nonconservation eq.

$$\boxed{\partial_\mu j^{\mu 5} = \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}}$$

has a global aspect.

In free fermion theory, the integral of the axial current conservation law gives

$$\int d^2x \, \partial_\mu j^{\mu 5} = N_R - N_L = 0. \quad (19.30)$$

In 2d QED, the conservation eq. for the axial current is replaced by the anomalous nonconservation eq. (19.18). If the r.h.s. of this eq. were the total derivative of a quantity falling off sufficiently rapidly at ∞, its integral should vanish, and $\epsilon^{\mu\nu} F_{\mu\nu}$ is a total derivative $\epsilon^{\mu\nu} F_{\mu\nu} = 2\partial_\mu (\epsilon^{\mu\nu} A_\nu)$.

$$\text{—— } (19.31).$$

Now, a world with a constant background electric field, the conservation law (19.30) must be violated. One way to see this is to note that the system gives a nonzero value to the Wilson line $\exp\left[-ie\int_0^L dx A_1(x)\right]$ —— (19.3

which forms a gauge-invariant closed loop due to the periodic boundary conditions.

Follow 3d Hamiltonian, in 1-d we have
(3.84)

$$H = \int dx\, \psi^\dagger (-i\alpha^1 D_1)\psi \qquad (19.33)$$

$$\alpha = \gamma^0\gamma^1 = \gamma^5$$

$$= \int dx \left\{ -i\psi_+^\dagger (\partial_1 - ieA^1)\psi_+ + i\psi_-^\dagger (\partial_1 - ieA^1)\psi_- \right\} \qquad (19.34)$$

The eigenstates of the covariant derivatives are wavefunctions $e^{ik_n x}$, with $k_n = \frac{2\pi n}{L}$, $n = -\infty \ldots \infty$ (19.35)

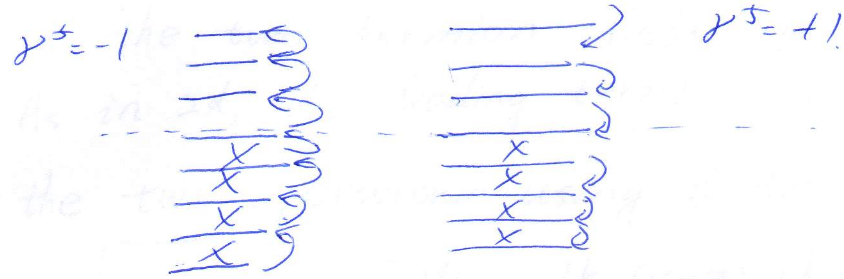
The single-particle eigenstates of H have energies

$$\psi_+ : \quad E_n = +(k_n - eA^1) \qquad (19.36)$$
$$\psi_- : \quad E_n = -(k_n - eA^1)$$

Now, adiabatically change the value of A^1. The fermion energy levels slowly shift in (19.36).

If A^1 change as $\Delta A^1 = 2\pi/eL$ (19.37) which brings the Wilson loop (19.32) back to its original value, the spectrum of H returns to its original form.

In this process, each level of H_+ moves down to the next position, and each level of H_- moves up to the next position, as follows:



The occupation numbers of levels should be maintained in this adiabatic process.

$\therefore$ One right-moving fermion disappears from the vacuum and one extra left-moving fermion appears, and $\int d^2x \left(\frac{e}{\pi}\epsilon^{\mu\nu}F_{\mu\nu}\right)$

$$= \int dt\,dx\, \frac{e}{\pi}\partial_0 A_1$$

$$= \frac{e}{\pi} L\,(-\Delta A^1) = -2 \quad \text{—— (19.3}$$

$$\Rightarrow\; N_R - N_L = \int d^2x \left(\frac{e}{\pi}\epsilon^{\mu\nu}F_{\mu\nu}\right)$$

(19.18) is satisfied!

The prescription is gauge invariant, but it leads to the nonconservation of the axial vector current.

19.2. The Axial Current in Four Dimensions.

⊙ The axial current operator equation.

* Compute the singular terms in the operator product of the two fermion fields in the limit $\epsilon \to 0$. As in 2d, the leading term is given by contracting the two operators using a free-field propagator.

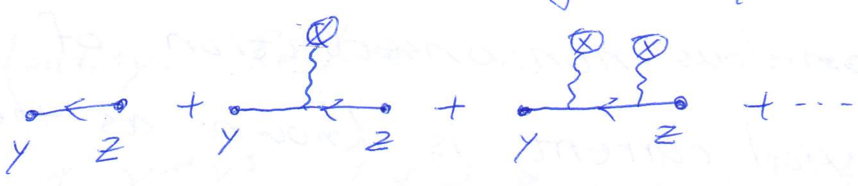
$$\Rightarrow \psi(y)\bar{\psi}(z) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(y-z)} \frac{i\not{k}}{k^2}$$

$$= -\not{\partial}\left(\frac{i}{4\pi^2}\frac{1}{(y-z)^2}\right)$$

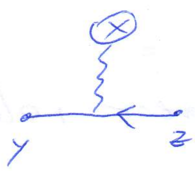
$$= \frac{-i}{2\pi^2}\frac{\gamma^\alpha}{(y-z)^4}(y-z)_\alpha \qquad (19.40)$$

As $(y-z) \to 0$ this will be highly singular, but it gives zero when traced with $\gamma^\mu\gamma^5$. To find the nonzero result, we should consider terms of higher order in the OPE.

In a nonzero background gauge field, the contraction of fermion fields is given by the series of diagram



where



$$= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} e^{-i(k+p)\cdot y} e^{ikz} \frac{i(\not{k}+\not{p})}{(k+p)^2}(-ie\not{A}(p))\frac{i\not{k}}{k^2} \qquad (19.41)$$

$$\Rightarrow \langle \bar{\psi}(x+\tfrac{\epsilon}{2}) \gamma^\mu \gamma^5 \psi(x-\tfrac{\epsilon}{2}) \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} e^{ik\cdot\epsilon} e^{-ip\cdot x} \, tr\left[(-\gamma^\mu\gamma^5) \frac{i(\not{k}+\not{p})}{(k+p)^2} (-ie\not{A}(p)) \frac{i\not{k}}{k^2}\right]$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} e^{ik\cdot\epsilon} e^{-ip\cdot x} \frac{4e\,\epsilon^{\mu\alpha\beta\gamma}(k+p)_\alpha A_\beta(p) k_\gamma}{k^2(k+p)^2} \quad (19.42)$$

For $\epsilon \longrightarrow 0$

$$\Rightarrow \quad \simeq 4e\,\epsilon^{\mu\alpha\beta\gamma} \int \frac{d^4p}{(4\pi)^4} e^{-ip\cdot x} p_\alpha A_\beta(p) \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot\epsilon} \frac{k_\gamma}{k^4}$$

$$= 4e\,\epsilon^{\alpha\beta\mu\gamma} (\partial_\alpha A_\beta(x)) \frac{\partial}{\partial \epsilon^\gamma} \left(\frac{i}{16\pi^2} \log \frac{1}{\epsilon^2}\right)$$

$$= 2e\,\epsilon^{\alpha\beta\mu\gamma} F_{\alpha\beta}(x) \left(\frac{-i}{8\pi^2} \frac{\epsilon_\gamma}{\epsilon^2}\right) \quad (19.43)$$

Substitute into (19.25)

$$\Rightarrow \partial_\mu j^{\mu 5} = \text{symm} \lim_{\epsilon \to 0} \left\{ \frac{e}{4\pi^2} \epsilon^{\alpha\beta\mu\gamma} F_{\alpha\beta} \left(\frac{-i\epsilon_\gamma}{\epsilon^2}\right) (-ie\epsilon^\nu F_{\mu\nu}) \right\}$$

for $\epsilon \to 0$.

$$\Rightarrow \partial_\mu j^{\mu 5} = \frac{-e^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} \quad (19.45)$$

express the anomalous nonconservation of the four-dim. axial current, is known as the Adler–Bell–Jackiw anomaly.

* Adler & Bardeen proved this operator relation is actually correct to all orders in QED perturbation.

⊙ Triangle Diagrams

Verify ABJ relation:

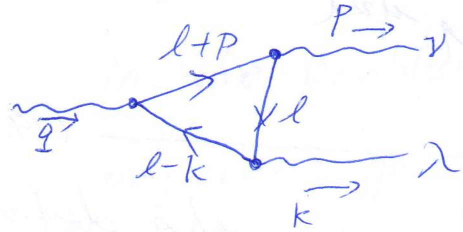
○ Analyze the matrix element

$$\int d^4x\, e^{-iq\cdot x} \langle p, k | j^{\mu 5}(x) | 0 \rangle$$

$$= (2\pi)^4 \delta^{(4)}(p+k-q)\, \epsilon^*_\nu(p)\, \epsilon^*_\lambda(k)\, \mathcal{M}^{\mu\nu\lambda}(p,k) \qquad \text{——— (19.46)}$$

Leading-order diagrams contributing to $\mathcal{M}^{\mu\nu\lambda}$

as follows:



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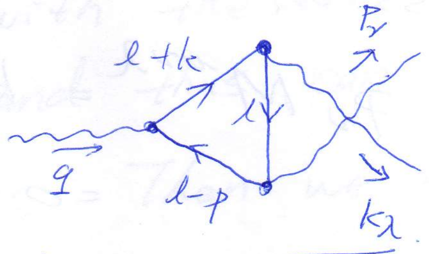


= first diagram with (p, ν) and (k,λ) interchange

$$\parallel$$

$$(-1)(-ie)^2 \int \frac{d^4 l}{(2\pi)^4} \mathrm{tr}\left[\gamma^\mu \gamma^5 \frac{i(\not{l}-\not{k})}{(l-k)^2} \gamma^\lambda \frac{i\not{l}}{l^2} \gamma^\nu \frac{i(\not{l}+\not{p})}{(l+p)^2}\right]$$

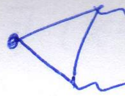
$$\text{——— (19.47)}$$

Operate the r.h.s. of (19.47) as to prove a Ward identity. Replace

$$q_\mu \gamma^\mu \gamma^5 = (\not{l} + \not{p} - \not{l} + \not{k})\gamma^5$$

$$= (\not{l}+\not{p})\gamma^5 + \gamma^5(\not{l}-\not{k}) \qquad \text{——— (19.48)}$$

(iq_μ)  $= e^2 \int \frac{d^4 l}{(2\pi)^4} \mathrm{tr}\left[\gamma^5 \frac{(\not{l}-\not{k})}{(l-k)^2} \gamma^\lambda \frac{\not{l}}{l^2} \gamma^\nu + \gamma^5 \gamma^\lambda \frac{\not{l}}{l^2} \gamma^\nu \frac{(\not{l}+\not{p})}{(l+p)^2}\right]$

$iq_\mu \cdot$  $= e^2\int \frac{d^4 l}{(2\pi)^4}\, tr\Big[\gamma^5 \frac{\not{l}}{l^2}\gamma^\lambda \frac{(\not{l}+\not{k})}{(l+k)^2}\gamma^\nu - \gamma^5 \frac{\not{l}}{l^2}\gamma^\nu \frac{(\not{l}+\not{p})}{(l+p)^2}\gamma$

(19.50)

This expression is manifestly antisymmetric under the interchange of (p,ν) and (k,λ), so the contribution of the 2^{nd} diagram in the diagrams cancels (19.47). Dimensional regularization of the diagrams will automatically insure the validity of the QED Ward identities for the photon emission vertices,

$$p_\nu M^{\mu\nu\lambda} = k_\lambda M^{\mu\nu\lambda} = 0. \quad (19.51)$$

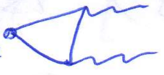
't Hooft and Veltman suggested using the definition

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (19.52)$$

in d dimensions. This def. has the consequence that γ^5 anticommutes with γ^μ for $\mu=0,1,2,3$ but commutes with γ^μ for other values of μ. In computing (19.47), the loop momentum l has components in all dimensions. Write

$$l = \underbrace{l_{\parallel}}_{\substack{\text{has nonzero components}\\ \text{in dim}=0,1,2,3.}} + \underbrace{l_{\perp}}_{\substack{\text{has nonzero components in the}\\ \text{order } d-4 \text{ dimensions.}}} \quad (19.53)$$

The terms involving the momentum shift P cancel,

then we derive iq_μ  $= e^2\left(\frac{-i}{2(4\pi)^2}\right) tr\left[2\gamma^5(-\not{k})\gamma^\lambda \not{p}\gamma^\nu\right]$

$$= \frac{e^2}{4\pi^2}\,\epsilon^{\alpha\lambda\beta\nu} k_\alpha P_\beta . \qquad (19.59)$$

This term is symmetric under the interchange of (p,ν) with (k,λ) $\therefore$ the 2nd diagram in Fig. 19.4 gives an equal contribution,

$$\langle p,k|\partial_\mu j^{\mu 5}(0)|0\rangle = \frac{-e^2}{2\pi^2}\,\epsilon^{\alpha\nu\beta\lambda}(-iP_\alpha)\,\epsilon^*_\nu(p)(-ik_\beta)\,\epsilon^*_\lambda(k)$$

$$= \frac{-e^2}{16\pi^2}\langle p,k|\,\epsilon^{\alpha\nu\beta\lambda} F_{\alpha\nu}F_{\beta\lambda}(0)|0\rangle . \qquad (19.60)$$

which is satify the expectation of ABJ anomaly equation.

◎ Chiral Transformation of the functional integral.
(Third way to check ABJ anomaly.)

1) Recall. fermionic functional integral.

$$Z = \int D\psi D\bar{\psi} \exp\left[i\int d^4x\, \bar{\psi}(i\not{D})\psi\right] \quad (19.61)$$

change variables

$$\psi(x) \longrightarrow \psi'(x) = (1 + i\alpha(x)\gamma^5)\psi(x)$$
$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = \bar{\psi}(1 + i\alpha(x)\gamma^5) \quad (19.62)$$

$$* \int d^4x\, \bar{\psi}'(i\not{D})\psi' = \int d^4x \left[\bar{\psi}(i\not{D})\psi - \partial_\mu \alpha(x)\bar{\psi}\gamma^\mu\gamma^5\psi\right]$$

$$= \int d^4x\left[\bar{\psi}(i\not{D})\psi + \alpha(x)\partial_\mu(\bar{\psi}\gamma^\mu\gamma^5\psi)\right] \quad (19.63)$$

<Def> $(i\not{D})\phi_m = \lambda_m\phi_m$, $\hat{\phi}_m(i\not{D}) = -iD_\mu\hat{\phi}_m\gamma^\mu$

$$= \lambda_m\hat{\phi}_m \quad (19.64)$$

$$\lambda_m^2 = k^2 = (k^0)^2 - (\vec{k})^2 \quad (19.65)$$

then expand ψ and $\bar{\psi}$

$$\Rightarrow \psi(x) = \sum_m a_m\phi_m(x)$$
$$\bar{\psi}(x) = \sum_m \hat{a}_m\hat{\phi}_m(x) \quad (19.66)$$

$a_m, \hat{a}_m$ are anticommuting coefficients

The functional measure over $\bar{\psi}, \psi$ can then be defined as $D\psi D\bar{\psi} = \prod_m da_m d\hat{a}_m \quad (19.67)$

If $\psi'(x) = (1 + i\alpha(x)\gamma^5)\psi(x)$, then ψ and ψ' are related by infinitesimal linear transformation $(1 + C)$, computed as follows:

$$a'_m = \sum_n \int d^4x\, \phi_m^\dagger(x)(1 + i\alpha(x)\gamma^5)\phi_n(x)\, a_n$$

$$= \sum_n (\delta_{mn} + C_{mn})\, a_n \quad \text{——— (19.68)}$$

$$D\psi' D\bar{\psi}' = \underbrace{J^{-2}}_{\text{Jacobian.}} D\psi D\bar{\psi} \quad \text{——— (19.69)}$$

$$\text{Jacobian} = \det(1+C) = \exp(\operatorname{tr}\log(1+C)) = \exp\Big[\sum_n C_{nn} + \cdots\Big] \quad \text{——— (19.70)}$$

$$\log J = i\int d^4x\, \alpha(x) \sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x) \quad \text{——— (19.71)}$$

looks like $\operatorname{tr}(\gamma^5) = 0$.

$$* \sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)$$

$$= \lim_{M\to\infty} \sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)\, e^{\lambda_n^2/M^2} \quad \text{——— (19.72)}$$

$$= \lim_{M\to\infty} \langle x|\operatorname{tr}\left[\gamma^5 e^{(i\not{D})^2/M^2}\right]|x\rangle \quad \text{— (19.7–}$$

* According to (16.107),

$$(i\not{D})^2 = -D^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} \quad \text{——— (19.74)}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$\Rightarrow \lim_{M\to\infty} \langle x| \operatorname{tr}\left[\gamma^5 e^{\left(-D^2 + \frac{e}{2}\sigma\cdot F\right)/M^2}\right]|x\rangle$$

$$= \lim_{M\to\infty} \operatorname{tr}\left[\gamma^5 \frac{1}{2!}\left(\frac{e}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}(x)\right)^2\right]\langle x| e^{\frac{-\partial^2}{M^2}}|x\rangle$$

$$= \lim_{x\to y}\int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} e^{k^2/M^2}$$

$$= i\int \frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/M^2}$$

$$= \frac{iM^4}{16\pi^2} \quad \text{——— (19.76)}$$

$$= \lim_{M\to\infty} \frac{-ie^2}{8\cdot 16\cdot \pi^2} M^4 \operatorname{tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma \frac{1}{(M^2)^2} F_{\mu\nu}F_{\lambda\sigma}(x)\right]$$

$$= \frac{-e^2}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}(x) \quad \text{——— (19.77)}$$

$$\Rightarrow J = \exp\left[-i\int d^4x\, \alpha(x)\left(\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}(x)\right)\right.$$

$$\text{——— (19.78)}$$

$$\Rightarrow Z = \int D\psi D\bar{\psi} \exp\left[i\int d^4x\left(\bar{\psi}(i\not{D})\psi + \alpha(x)\left\{\partial_\mu j^{\mu 5} + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}\right\}\right)\right] \quad \text{——— (19.79)}$$

Varying the exponent w.r.t. $\alpha(x)$, we derive ABJ anomaly eq.

* If d is even, we can construct a matrix γ^5 which is anticommutes with all of the Dirac matrices by taking their product. Then the functional derivation leads straightforwardly to the result

$$\partial_\mu j^{\mu 5} = (-1)^{n+1} \frac{2e^n}{n!(4\pi)^n} \epsilon^{\mu_1 \mu_2 \cdots \mu_{2n}} F_{\mu_1 \mu_2} \cdots F_{\mu_{2n-1} \mu_{2n}}$$

$(n = d/2)$. ——— (19.80)

19.4. Chiral Anomalies and Chiral Gauge Theories.

For a theory contains massless Dirac fermions ψ_i, we can write the kinetic energy term in the helicity basis such as

$$\mathcal{L} = \psi_{Li}^{\dagger} i\bar{\sigma}\cdot\partial\psi_{Li} + \psi_{Ri}^{\dagger} i\sigma\cdot\partial\psi_{Ri} \quad \text{— (19.120)}$$

If assign the left-handed fields to a representation r of G and take the right-handed fields to be invariant under G.

$$\Rightarrow \mathcal{L} = \psi_{Li}^{\dagger} i\bar{\sigma}\cdot D\psi_{Li} + \psi_{Ri}^{\dagger} i\sigma\cdot\partial\psi_{Ri} \quad \text{— (19.121)}$$

with $D_\mu = \partial_\mu - igA_\mu^a t_r^a$. In more conventional notation, (19.121) becomes

$$\mathcal{L} = \bar{\psi} i\gamma^\mu\left(\partial_\mu - igA_\mu^a t_r^a\left(\frac{1-\gamma^5}{2}\right)\right)\psi \quad \text{— (19.122)}$$

It's straight-forward to verify (19.122) is invariant to the local gauge transformation

$$\begin{aligned} \psi &\to \left(1 + i\alpha^a t^a\left(\frac{1-\gamma^5}{2}\right)\right)\psi \\ A_\mu^a &\to A_\mu^a + \frac{1}{g}\partial_\mu\alpha^a + f^{abc}A_\mu^b\alpha^c \end{aligned} \quad \text{— (19.123)}$$

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Central Idea: gauge field couples only to the left-handed fermions.

$\Rightarrow$ Assigning the left-handed components of quarks and leptons to doublets of an $SU(2)$ gauge symmetry $Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$, $L_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$

It's useful to rewrite the Lagrangian for the right-handed fermions as.

$$\int d^4x\, \psi^{\dagger}_{R_i} i\sigma\cdot\partial \psi_{R_i} = \int d^4x\, \psi'^{\dagger}_{L_i} i\bar{\sigma}\cdot\partial \psi'_{L_i} \quad \text{——(19.126)}$$

where: $\psi'_{L_i} = \sigma^2 \psi^*_{R_i}$, $\psi'^{\dagger}_{L_i} = \psi^T_{R_i}\sigma^2$. ——(19.125)

Note: if fermions are coupled to gauge fields in the representation r, this manipulation changes the covariant derivative as follows:

$$\psi^{\dagger}_R i\sigma\cdot(\partial - igA^a t^a_r)\psi_R = \psi'^{\dagger}_L i\bar{\sigma}\cdot(\partial + igA^a (t^a_r)^T)\psi'_L$$

$$= \psi'^{\dagger}_L i\bar{\sigma}\cdot(\partial - igA^a t^a_{\bar{r}})\psi'_L \quad \text{—(19.12-}$$

↑

belong to conjugate representation to r.

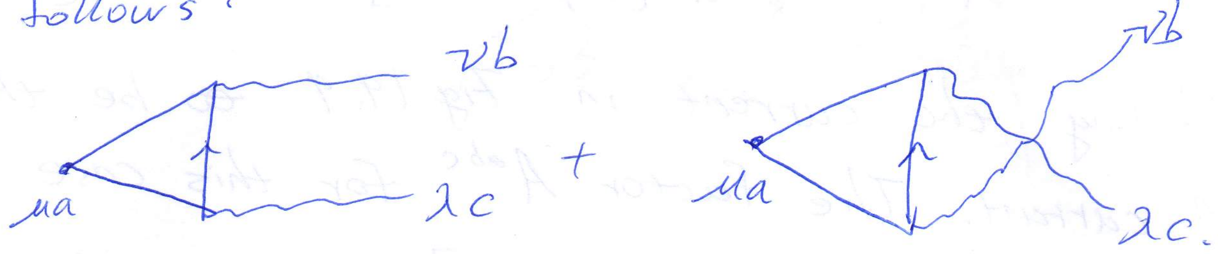
* Rewriting a system of Dirac fermions leads to $R = r \oplus \bar{r}$, a real representation in the sense described in $t^a_{\bar{r}} = -(t^a_r)^* = -(t^a_r)^T$.

The rewriting (19.125) transforms the mass term of the QCD Lagrangian as follows:

$$m\bar{\psi}_i\psi_i = m(\psi^{\dagger}_R\psi_L + h.c.) = -m(\psi'^{T}_{L_i}\sigma^2\psi_{L_i} + h.c.) \quad \text{—(19.}$$

which is the form of the Majorana mass term. The most general form that can be buit purely from left-handed fermion fields is $\Delta\mathcal{L}_M = M_{ij}\psi'^{T}_{L_i}\sigma^2\psi_{L_i} + h.c.$ (1

In a gauge theory of left-handed massless fermions, consider computing the diagram as follows:



in which the external fields are non-Abelian gauge bosons and the marked vertex representations the gauge symmetry current

$$j^{\mu a} = \bar{\psi}\gamma^{\mu}\left(\frac{1-\gamma^5}{2}\right)t^a\psi \quad (19.130)$$

If we regularize the diagram as in sec 19.2, the term containing a γ^5 has an axial vector anomaly that leads to the relation

$$\langle p, \nu, b; k, \lambda, c|\partial_\mu j^{\mu a}|0\rangle$$
$$= \frac{g^2}{8\pi^2}\epsilon^{\alpha\nu\beta\lambda}p_\alpha k_\beta \underbrace{A^{abc}}_{\text{a trace}} \quad (19.131)$$

a trace over group matrices in the representation R:

$$A^{abc} = tr\left[t^a\{t^b, t^c\}\right] \quad (19.132)$$

* Example:

Consider the proto-type weak interaction gauge theory in (19.124). If two gauge bosons in Fig 19.9 are SU(2) gauge bosons and the current $j^{\mu a}$ is an SU(2) gauge current, we would evaluate (19.132) by substituting $t^a = \tau^a = \sigma^a/2$ and using the relation $\{\sigma^b, \sigma^c\} = 2\delta^{bc}$.

$$\therefore \; A^{abc} = \frac{1}{8} tr\left[\sigma^a \cdot 2\delta^{bc}\right] = 0 \quad \text{——} (19.133)$$

By taking the current in Fig 19.9 to be the EM-current. The factor A^{abc} for this case is

$$tr\left[Q\{\tau^b, \tau^c\}\right], \quad \text{——} (19.134)$$

Q is the matrix of electric charges.

If simplify as in (19.133), the trace (19.134) becomes $\frac{1}{2} tr[Q]\delta^{bc}$ —— (19.135)

And, if sum over one quark doublet and one lepton doublet, with a factor 3 for colors,

$$\Rightarrow tr[Q] = 3 \cdot \left(\frac{2}{3} - \frac{1}{3}\right) + (0-1) = 0. \quad \text{——} (19.136)$$

If the fermion representation R is real, R is equivalent to its conjugate representation $\bar{R}$.

$\therefore$ t_R^a is related by a unitary transformations to $t_{\bar{R}}^a = -(t_R^a)^T$ $\therefore$ (19.132) is invariant to unitary transformations of the t^a, we can replace t_R^a by $t_{\bar{R}}^a$. Then

$$A^{abc} = tr\left[(-t^a)^T \{(-t^b)^T, (-t^c)^T\}\right]$$

$$= -tr\left[\{t^c, t^b\} t^a\right] = -A^{abc} \quad \text{——} (19.137)$$