NOTES ON LEARNING HOM-POLYTOPES

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Contents

1. Affine spaces and Affine transformations

Let V be a $\mathbb{R}$-vector space.

Definition 1. *A is an affine subset of V, if $A \subset V$, $A = v + U$ for some $v \in V$, $U \subset V$.*

Definition 2. *For $v \in V$ and U a subspace of V, the affine subset $v + U$ is said to be parallel to U.*

Definition 3. *Let U be a subspace of V. Then the quotient space V/U is the set of all affine subsets of V parallel to U. In other words,*

$$V/U = \{v + U : v \in V\}.$$

Addition and scalar multiplication on V/U:

Definition 4. *Let U be a subspace of V. Then addition and scalar multiplication are defined on V/U by*

$$(v + U) + (w + U) = (v + w) + U$$

$$\lambda(v + U) = (\lambda v) + U$$

for $v, w \in V$ and $\lambda \in \mathbb{R}$.

Definition 5. *Let $lin(X) = \{a_1(x_1 - x_0) + ... + a_m(x_m - x_0) | n \in \mathbb{N}, x_0, x_1, ..., x_m \in X, a_1, ..., a_m \in \mathbb{R}\}$, then $lin(X)$ is the linear hull of X.*

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Definition 6. *Let $X \subset V$, then aff$(X) := \{a_1(x_1) + ... + a_m(x_m) | n \in \mathbb{N}, x_1, ..., x_m \in X, a_1, ..., a_m \in \mathbb{R}, a_1 + ... + a_m = 1\}$, and aff$(X)$ is called the affine hull of X.*

Property 1. *Let $A \neq \varnothing$. $A \subset V$ is affine, if and only if $\lambda v + (1 - \lambda)w \in A$, $\forall v, w \in A$, and $\forall \lambda \in \mathbb{R}$, i.e., A is closed under affine linear combinations.*

Proof.
"$\Rightarrow$" Given A as an affine subset of V. Assume $A = x + U$, $U \subset V$, where $x \in V$. Let $v = x + u_1 \in A$, and $w = x + u_2 \in A$. We also know $(\lambda u_1 + (1 - \lambda)u_2) \in U$, since U is also a vector space.

Claim: $(\lambda v + (1 - \lambda)w) \in A$.

Since $(\lambda v + (1 - \lambda)w) = \lambda(x + u_1) + (1 - \lambda)(x + u_2) = \lambda x + \lambda u_1 + x + u_2 - \lambda x - \lambda u_2 = x + (\lambda u_1 + u_2 - \lambda u_2) = x + (\lambda u_1 + (1 - \lambda)u_2) \in A$. where $(\lambda u_1 + (1 - \lambda)u_2) \in U$. It follows that $(\lambda v + (1 - \lambda)w) \in A$.

Furthermore, in general, we have the following:
Since U is a vector subspace of V, so U is closed under linear combinations. Hence, $\forall \lambda_i \in \mathbb{R}$, we have $\sum_{i=1}^n \lambda_i u_i \in U$. If we let $v_i = x + u_i$, where $x \in A$, $u_i \in U$, $i \in \{1, 2, ..., n\}$, and $\sum_{i=1}^n \lambda_i = 1$, then we have $\sum_{i=1}^n \lambda_i v_i = \sum_{i=1}^n (x + u_i) = \sum_{i=1}^n \lambda_i x + \sum_{i=1}^n \lambda_i u_i = x + \sum_{i=1}^n \lambda_i u_i \in x + U$.

"$\Leftarrow$" Let $(\lambda v + (1 - \lambda)w) \in A$, $\forall v, w \in A$, and $\forall \lambda \in \mathbb{R}$ be given. Want to show $A \subset V$ is affine.

First proof.
Let's prove a more general case that we are given more than two elements of A.
Let $\sum_{i=1}^n \lambda_i x_i$ be given as an affine linear combination, where $x_i \in A$.
$\Rightarrow \sum_{i=1}^n \lambda_i x_i = v + (\sum_{i=1}^n \lambda_i x_i) - (\sum_{i=1}^n \lambda_i) v = v + (\sum_{i=1}^n \lambda_i (x_i - v)) \in v + U$. Then by the first definition, i.e., the definition of affine subset, A must be affine, and this completes the proof.

Second proof.
If we start with defining affine hull, and then proving aff$(A) = A$, (since A is affine, and affine hull is the smallest affine set containing A, so it contains and is equal to itself) then we have one condition for any element, $a \in A$, in affine set A that it can be represented as an affine linear combination, $a = \sum_{i=1}^n \lambda_i x_i$, where $\sum_{i=1}^n \lambda_i = 1$, $\forall \lambda_i \in \mathbb{R}$, and

$x_i \in A$. Then, we can have the following second proof for this direction. However, if we didn't prove that first, then we don't have that property yet. Because one common approach to prove it is based on the theorem that we are proving, so it's important to notice this to avoid a circular logic, and that's why we have the first proof. Hence, if we already proved that $\text{aff}(A) = A$ by using the idea in the first prove, then we can also prove this direction in the following way: To show A is closed under affine linear combinations.
Furthermore, this is also equivalent to show $A - \alpha$ is a vector subspace of V where $\alpha \in A$.
That is, we can pick any $\alpha \in A$, and since $-\alpha + A$ is a vector subspace of V, we check the following properties of vector space:

1. $0 \in -\alpha + A$, since $\alpha \in A$.

2. we want to check $-\alpha + A$ is closed under scalar multiplication.
Since, $\forall \lambda \in \mathbb{R}, \forall a \in A$, we have $\lambda(-\alpha + a) = -\lambda\alpha + \lambda a = -\alpha + (\alpha - \lambda\alpha) + \lambda a = -\alpha + ((1-\lambda)\alpha + \lambda a) \in -\alpha + A$, hence $\lambda(-\alpha + a) \in -\alpha + A$.

3. Want to show: $-\alpha + A$ is closed under addition.
Claim: $-\alpha + a, -\alpha + b \in -\alpha + A$, and $\alpha, a, b \in A$, we check $(-\alpha + a) + (-\alpha + b) \in A$.
Since $(-\alpha + a) + (-\alpha + b) = -\alpha + (-\alpha + a + b)$ where the coefficients of $(-\alpha + a + b)$ is $(-1 + 1 + 1) = 1$.
Alternatively, since we already proved $-\alpha + A$ is closed under scalar multiplication, so we also have $(-\alpha + a) + (-\alpha + b) = 2\left(\frac{a}{2} + \frac{b}{2} - \alpha\right) \in -\alpha + A$, and the sum of these coefficients is $\frac{1}{2} + \frac{1}{2} - 1 = 1$.

$\square$

Additionally, by Definition 1, we have $\text{aff}(A) = A = v + U = v + \text{lin}(A)$. Therefore, in general, we can summarize the above proof, and rewrite the above property into the following theorem:

Theorem 7. *A subset A of V is an affine subspace if and only if it is closed under affine linear combinations.*

A linear combination can be seen as an affine combination by substituting 0 with the coefficient $1 - \sum_i \lambda_i$. Additionally, affine combinations are also called barycentric combinations, and $\lambda_i' s$ are barycentric coordinates.

Property 2. *Let $v_i \in V$, $A = \{\sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R},$ and $\sum_{i=1}^{n} = 1\}$. Then, A is an affine subset of V.*

Proof. Let $u = \sum_{i=1}^{n} \lambda_i v_i \in A$, and $v = \sum_{i=1}^{n} \gamma_i v_i \in A$. Substitute u and v into the first property, or by Thm 6, then we obtain: $\lambda u + (1-\lambda)v = \lambda(\sum_{i=1}^{n} \lambda_i v_i) + (1-\lambda)(\sum_{i=1}^{n} \gamma_i v_i) = \sum_{i=1}^{n} (\lambda\lambda_i + (1-\lambda)\gamma_i)$. However, $\sum_{i=1}^{n} (\lambda\lambda_i + (1-\lambda)\gamma_i) = \lambda \sum_{i=1}^{n} \lambda_i + (1-\lambda) \sum_{i=1}^{n} \gamma_i = \lambda + (1-\lambda) = 1$. Hence, by Thm 6, or the first property, A is affine, and $A \subset V$. □

Property 3. *Let $v_i \in V$, $A = \{\sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R},$ and $\sum_{i=1}^{n} \lambda_i = 1\}$. Then, $A = v + U$ where v and U are defined as in Definition 1.*

To prove $A = v + U$, since we have proved for each affine linear combination, i.e., given in the form as the element of the set A, it has the form: $\sum_{i=1}^{n} \lambda_i x_i \in v + U$ in Thm 6, so we have proved the "$\subseteq$" direction. Now, let's prove the other direction.

Proof. Take $\sum_{i=1}^{n} \gamma_i \cdot x_i \in U$, where U is a vector subspace of V. $\exists k \in \mathbb{R}$, and $x_i \in A$ such that $\gamma_i = k(x_i - v)$. Thus, $v + \sum_{i=1}^{n} \gamma_i x_i = v + \sum_{i=1}^{n} \gamma_i (x_i - v) = v + (\sum_{i=1}^{n} \gamma_i x_i) - (\sum_{i=1}^{n} \gamma_i) \cdot v \in A$. The last equality holds, since by the given definition of elements in A, the summation of coefficients is $1 + (\sum_{i=1}^{n} \gamma_i) - (\sum_{i=1}^{n} \gamma_i) = 1$. This completes the proof. □

By the above property, we also have:

Property 4. $\dim(A) = dim(v + U) = dim(U) \leq n - 1$.

Proof. The first equality is obvious, since it's by the definition of affine subset. The second property is also obvious since $v \in A$. For the third inequality, without loss generality, we take $v = v_k$ in Definition 1, where $k \in \{1, ..., n\}$. The idea of the proof is to prove both sides contain each other:

$$A = v_k + \operatorname{span}\{v_2 - v_k, v_3 - v_k, ..., v_n - v_k\}.$$

Hence, $\dim(A) \leq n - 1$. □

Property 5. *Let $v_i \in V$, $A = \{\sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R},$ and $\sum_{i=1}^{n} = 1\}$. Then, every affine subset of V that contains v_i, $i \in \{1, ..., n\}$, also contains A.*

Proof. In property 3, we have proved $A = v + U$, in other words,

$$\begin{aligned} A &= \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R}, \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\} \\ &= v + U \end{aligned}$$

and by Definition 1, we know A is affine. Now, since we also have defined affine hull, and it's clearly that by Definition 6, the given set A is an affine hull. Thus, we have A=aff$(A) = v + U$, and hence aff(A) is the smallest affine set contains A. Therefore, every affine subset of V that contains all $v_i's$ also contains A. □

Proposition 1. *A set $X \subseteq \mathbb{R}^n$ is convex, if and only if $\sum_{i=1}^n \lambda_i x_i \in X$, $\forall n \in \mathbb{N}$, $x_i \in X$, and $0 \leq \lambda_i \leq 1$, with $\sum_{i=1}^n \lambda_i = 1$.*

Linear combinations in this proposition is called convex linear combinations.

Definition 8. *The convex hull, conv(X) of a set $X \subseteq \mathbb{R}^d$ is the intersection of all convex supersets of X.*

Property 6. *For arbitrary X, the convex hull of X is the union of the convex hulls of the finite subsets of X.*

Definition 9. *A subset X of V is convex if contains a line segment $[x, y] = \{ax + (1-a)y : 0 \leq a \leq 1\}$ for all points $x, y \in X$. Open or halfopen line segments (x, y) and $(x, y]$ are defined respectively.*

Since every linear map f must send the zero vector to the zero vector, i.e., $f(0) = 0$. Nevertheless, $\forall u \in V$, u is nonzero, the function $T_u(x) = x + u, \forall x \in V$ is what we usually have to confront in applications. This type of functions are called translations, and they are not linear for $u \neq 0$, because $T_u(0) = 0 + u = u$.

Functions combining linear maps and translations are what we want to pay attention to. Hence, it's good to know more about their properties.

The conditions in affine combinations that $\sum_{i=1}^n \lambda_i$ ensures that affine combinations are preserved under translations. Consider $f : V \to W$, where V and W are two vector spaces, such that there is some linear map $T : V \to W$ and some fixed vector $b \in V$ (a translation vector), hence we have $f(x) = T(x) + b, \forall x \in V$. The functions of this type preserve affine combinations.

Proposition 2. *The function f preserves affine combinations: $f : V \to W$, $f(x) = T(x) + b, \forall x \in V$, where $T : V \to W$ is a linear map and b is some fixed vector in V, then for every affine combination $\sum_{i=1}^n \lambda_i u_i$, where $\sum_{i=1}^n \lambda_i = 1$, we have*

$$f\left(\sum_{i=1}^n \lambda_i u_i\right) = \sum_{i=1}^n \lambda_i f(u_i).$$

Proposition 3. *For any two vector space V, W, let $f : V \to W$ be any function that preserves affine combinations, then for any $h \in V$, the function $T : V \to W$ given by $T(x) = f(x+h) - f(h)$ is a linear map independent of h, and $f(x+h) = f(x) + f(h)$, $\forall x \in V$. In particular, for $h = 0$, if we let $h = f(0)$, then $f(x) = T(x) + h$ for all $x \in V$.*

The point h can be seen as a chosen origin in V, hence the function f maps the origin $h \in V$ to the origin in $f(h) \in f(V)$. Hence, it is natural to define affine map as follows:

Definition 10. *For any two vector spaces V, W, a function $f : V \to W$ is an affine map if f preserves affine linear combinations, i.e., for every affine linear combination $\sum_{i=1}^{n} \lambda_i u_i$, where $\sum_{i=1}^{n} \lambda_i = 1$, we obtain*

$$f\left(\sum_{i=1}^{n} \lambda_i u_i\right) = \sum_{i=1}^{n} \lambda_i f(u_i).$$

Equivalently, a function $\alpha : V \to W$ is an affine map if there is some linear map $T : V \to W$ and some fixed vector $h \in V$ such that $\alpha(x) = T(x) + h, \forall x \in V$. Notice that a linear map always maps the standard origin $0 \in V$ to the standard origin $0 \in W$, and this is not always the case for an affine map (unless we take $h = 0$). Notice that $T(x)$ is also called a (unique) linear form on V, and the space of linear forms on V is denoted by $V*$.

We can use affine forms to define the following new notions based on natural topologies on $V \backsimeq \mathbb{R}^d$ and $\mathbb{R}$.

2. Polyhedra and their faces

Definition 11. *A open halfspace, $H_\alpha^>$, is defined by nonconstant affine forms α, for all $x \in V$ such that $\alpha(x)$ is positive.*

$$H_\alpha^> = \{x \in V : \alpha(x) > 0\}.$$

A closed halfspace, $H_\alpha^>$, is defined by nonconstant affine forms α, for all $x \in V$ such that $\alpha(x)$ is nonnegative.

$$H_\alpha^+ = \{x \in V : \alpha(x) \geq 0\}.$$

A hyperplane is defined as

$$H_\alpha = \{x \in V : \alpha(x) = 0\},$$

Likewise, by symmetry, we also have: $H_\alpha^< = H_{-\alpha}^>$, and $H_\alpha^- = H_{-\alpha}^+$.

Definition 12. *A subset $P \subset V$ is called a polyhedron if it is the intersection of finitely many closed halfspaces. The dimension of P is given by $\dim aff(P)$. A d-polyhedron has dimension d.*

A polytope is a bounded polyhedron. A 2-polytope is called a polygon.

A morphism of polyhedroa P and Q is a map $\phi : P \to Q$ that can be extended to an affine map $\phi' : \mathrm{aff}(P) \to \mathrm{aff}(Q)$.

For simplicity, $H_{\alpha_i}^+$ is usually written as H_i^+, assuming that a suitable affine form α_i has been chosen to define the halfspace H_i^+.

Proposition 4. *Let $P = H_1^+ \cap ... \cap H_m^+$ be a polyhedron. Then $aff(P)$ is the intersection of those hyperplanes $H_i, i = 1, ..., m$, that contain P.*

Our next goal is to describe the face structure for polyhedra of arbitrary dimension.

Definition 13. *A hyperplane H is called a support hyperplane of the polyhedron P if P is contained in one of the two closed halfspaces bounded by H and $H \cap P \neq \varnothing$.*

A facet of P is a face of dimension $\dim P - 1$. The polyhedron P itself and $\varnothing$ are the imporper faces of P.

Proposition 5. *Let $P \subset V$ be a polyhedron, and H a hyperplane such that $H \cap P \neq \varnothing$. Then H is a support hyperplane associated with a proper face of P if and only if $H \cap P \subset \partial P$.*

Lemma 1. *Suppose $P = H_1^+ \cap ... \cap H_n^+$ is a polyhedron. Then a convex set $X \subset \partial P$ is contained in H_i for some i.*

Theorem 14. *Let $P \subset V$ be a polyhedron such that $d = \dim P = \dim V$. Then the halfspaces $H_1^+, ..., H_n^+$ in an inrredundant representation $P = H_1^+ \cap ... \cap H_n^+$ are uniquely determined. In fact, the sets $F_i = P \cap H_i, i = 1, ..., n$, are the faces of P.*

Corollary 1. *Let P be a polyhedron.*
(a) Then ∂P is the union of the facets of P.
(b) Each proper face of P is contained in a facet.

Proposition 6. *Let F be a face of the polyhedron P and $G \subset F$. Then G is a face of P if and only if it is a face of E.*

3. Finite generation of cones

Proposition 7. *For two finitely generated pointed cones C_1 and C_2, the set of linear maps $C_1 \longrightarrow C_2$ is a finitely generated cone in the appropriate ambient vector space.*

Proof. Let $I = \{1, 2, ..., m\}$, and $J = \{1, 2, ..., n\}$.
$C_1 = \sum_{i \in I} \mathbb{R}_+ x_i \subset V \subset \mathbb{R}^m, x_i \in C_1$.

□

Proposition 8. *For two finitely generated pointed cones C_1 and C_2, the set of linear maps $C_1 \longrightarrow C_2$ is a finitely generated cone in the appropriate ambient vector space.*

Proof. Let $I = \{1, 2, ..., m\}$, and $J = \{1, 2, ..., n\}$. We also let$C_1 = \sum_{i \in I} \mathbb{R}_+ x_i \subset V \subset \mathbb{R}^m, x_i \in C_1$, and $\dim C_1 = \dim V = m$.

On the other hand, for the target set, we have $C_2 = \cap_{i \in J} H^+_{\alpha_j} \subset W \subset \mathbb{R}^n$, and $\dim C_2 = \dim W = n$.

We then define the set of all linear maps from C_1 to C_2 as follows:

$$Lin(C_1, C_2) = \{f : C_1 \longrightarrow C_2, f \text{ is linear}\}.$$

Then we can denote the ambient space as $Lin(C_1, C_2) \subset Lin(V, W)$. Next, let's define an evaluation map ψ_{α_j, x_i}

$$Lin(C_1, C_2) \xrightarrow{\phi_{\alpha_j, x_i}} \mathbb{R}_+$$

$$f \xmapsto{\phi_{\alpha_j, x_i}} \alpha_j(f(x_i))$$

That is, $\phi_{\alpha_j}(f(x_i)) = (\phi_{\alpha_j} \circ f)(x_i) = \alpha_j(f(x_i))$.

Further, we know $\alpha_j(f(x_i)) \geq 0$, where $x_i \in C_1$, $f(x_i) \in C_2$, $f \in Lin(C_1, C_2)$, and $\alpha_j(f(x_i)) \in \mathbb{R}_+$. Hence, we have

$$Lin(C_1, C_2) = \cap_{i \in I, j \in J} \{f | \phi_{\alpha_j, x_i(f) \geq 0}\}$$

$$= \cap_{i \in I, j \in J} \{f(x_i) \in C_2 | \alpha_j(f(x_i)) \geq 0, where x_i \in C_1\}$$

$$:= \cap_{i \in I, j \in J} H^+_{\alpha_j, x_i}.$$

Therefore, by the definition of cones, $Lin(C_1, C_2)$ is a cone The above equality will be proved in a more general case $Hom(P,Q) = \cap_{i\in I, j\in J} H^+_{\alpha_j, x_i}$ in the following. □

Theorem 15. *The set $Lin(C_1, C_2)$ is pointed.*

Proof. Assume to the contrary, $y \neq 0$, and $y, -y \in Lin(C_1, C_2)$. Take $\forall x \in C_1$. Then $y(x) \in C_2$, $-y(x) \in C_2$ by the definition of $Lin(C_1, C_2)$. However, this contradicts to the fact that C_2 is also pointed. □

Definition 16. *Define $Hom(P,Q) = \{f : P \to Q | \ f \text{ is affine. } \}$ is a polytope in an appropriate ambient space. Also, Define $\triangle_n = conv\ (0, e_1, e_2, ..., e_n)$.*

Theorem 17. *The set $Lin(\mathbb{R}^n_+, C) \cong C^n$.*

Proof. Define an evaluation map ϕ:

$$Hom_{Lin}(\mathbb{R}^n_+, C) \xrightarrow{\quad\phi\quad} C^n$$

$$f \xmapsto{\quad\phi\quad} (f(e_1), f(e_2), ..., f(e_n))$$

Since $\phi(f+g) = ((f+g)(e_1), (f+g)(e_2), ..., (f+g)(e_n))$, and $\phi(f) + \phi(g) = (f(e_1), f(e_2), ..., f(e_n)) + (g(e_1), g(e_2), ..., g(e_n)) = ((f+g)(e_1), (f+g)(e_2), ..., (f+g)(e_n)) = \phi(f+g)$. Likewise, we have $\phi(f \cdot g) = \phi(f) \cdot \phi(g)$.

To show this map is injective, since linear maps f and g are uniquely determined by basis (they are restricted maps), so ϕ is injective.

To show this map is surjective we take $X \in C^n$, then for some $x_j \in C$, we have

$$X = (x_1, x_2, ..., x_n)\,.$$

There is $f \in Hom_{Lin}(\mathbb{R}^n_+, C)$ such that

$$\phi(f) = X = (x_1, x_2, ..., x_{n+1}) = (f(e_1), ..., f(e_n)).$$

Therefore, for all $X \in C^{n+1}$, there exists $f \in Hom(\mathbb{R}^n_+, C)$ such that $f(e_j) = x_j$.

□

4. Hom-polytopes

Theorem 18. *The set $Hom(\triangle_n, P) \cong P^{n+1}$ is pointed.*

Proof. This follows the fact that $\dim(Hom(\triangle_n, P)) = \dim(\triangle_n)\dim(P) + \dim(P)$. Hence in $Hom(\triangle_n, P)$, we have v_i, $i \in \{1, 2, ..., n+1\}$ vertices. Then for any affine map $f : \triangle_n \to P$ is uniquely defined by the basis and its restriction map:

$$f|_{vert(\triangle_n)}.$$

Then, we can define the evaluation map (which is a mutually inverse affine map) as follows:

$$Hom(\triangle_n, P) \xrightarrow{\phi} P^{n+1}$$

$$f \xmapsto{\phi} \Pi_{i=1}^{n+1} f(v_i)$$

and

$$P^{n+1} \xrightarrow{\phi^{-1}} Hom(\triangle_n, P)$$

$$\Pi_{j=1}^{n+1} x_j \xmapsto{\phi^{-1}} z$$

Since any map $vert(\triangle_n) \to P$ extends uniquely to an affine map $\triangle_n \to P$. Hence, the mutually inverse affine map could be derived as follows

$$\phi^{-1} : \Pi_{j=1}^{n+1} x_j \mapsto f$$

where this f satisfies the restriction condition $f(v_i) = x_i$ so it is uniquely defined on $\triangle_n$.

$\square$

Lemma 1. *If affine spaces $\mathbb{A} \subset V, \mathbb{A}' \subset V'$, then $aff(\mathbb{A}, \mathbb{A}')$ should be an affine subspaces of vector space $aff(V, V')$.*

Proof. Clearly, $aff(V, V')$ is a vector space ($0 \in aff(V, V')$, closed under addition and scalar multiplication).
Let $\mathbb{A} = U + x$ and $\mathbb{A}' = W + y_0$, $U \subset V$, $W \subset V'$. Let $\pi_U : V \to U$ be linear projection mapping, and $\pi : V \to \mathbb{A}$ be the affine projection ($\pi^2 = \pi$).

Let $\theta : \mathbb{A}' \to V'$ be the identity embedding, then the embedding

$$aff(\mathbb{A}, \mathbb{A}') \xrightarrow{\phi} aff(V, V')$$

$$f \xmapsto{\phi} \theta \circ f \circ \pi$$

makes $aff(\mathbb{A}, \mathbb{A}')$ into an affine subspace of $aff(V, V')$. □

Theorem 19. *If affine spaces* $\mathbb{A} \subset V, \mathbb{A}' \subset V'$*, then* $aff(\mathbb{A}, \mathbb{A}')$ *should have dimension* $\dim(aff(\mathbb{A}, \mathbb{A}')) = \dim(\mathbb{A}) \dim(\mathbb{A}') + \dim(\mathbb{A}')$.

Proof. Let $Lin(U, W) = \{g : U \to W, \}$, $\mathcal{B}$ be the basis of $Lin(U, W)$, $\mathcal{C}$ be the basis of W, and Ω be the vector space spanned by $\mathcal{B} \cup \mathcal{C}$ where $y_0 \notin Lin(U, W)$.
We also let

$$y_0 \in V',$$

$$W \subset V',$$

$$\mathbb{A}' = y_0 + W,$$

and finally

$$f(u) \in \mathbb{A}' = W + y_0.$$

We want to show: $\Omega + y_0 = aff(\mathbb{A}, \mathbb{A}')$

- "$\subseteq$" Let $f(u) + f(w + y_0) \in \Omega + y_0$. f extends to a linear map: $V \to V'$. $w + y_0 \in V'$. It follows that $f(u) + f(w + y_0) \in aff(\mathbb{A}, \mathbb{A}')$.
- "$\supseteq$" Let $f(x) + k \in aff(\mathbb{A}, \mathbb{A}') \Rightarrow f(x) + k$ is the restriction of an affine map $V \to V'$.

  $$f(x) + k = f(u + x_0) + k = f(u) + f(x_0) + k.$$

  Since $f(0) = 0$, $f(x_0) + k \in \mathbb{A}'$. Hence $f(x) + k \in \Omega + y_0$.
  Resulting form the notion of linear algebra: $\dim(Lin(V, W)) = \dim(U) \dim(W) = \dim(\mathbb{A}) \dim(\mathbb{A}')$.

Therefore, $\#(\mathcal{B} \cup \ \mathcal{C}) = \dim(\mathbb{A}) \dim(\mathbb{A}') + \dim(\mathbb{A}')$, as promised.

□

Theorem 20. $\dim(Hom(P, Q)) = \dim(P) \dim(Q) + \dim(Q)$.

Proof. Since the interior of the Hom-polytope $int(Hom(P,Q)) = \cap_{\alpha_j,x_i}\{f(x_i \in Q)|\alpha_j(f(x_i) > 0)\} = \cap_{\alpha_j,x_i}\{f \in aff(P,Q)|\phi_{\alpha_j}(f(x_i) > 0)\}$ is nonempty and open, hence it's fully dimentional. Based on the above result, since $Hom(P,Q)$ is full-dimensional, hence

$$\dim(Hom(P,Q)) = \dim(aff(P,Q)) = \dim(P)\dim(Q) + \dim(Q).$$

□

Definition 21. *Let P and Q are polytoptes, i.e., they are convex sets. Then $Hom(P,Q) := \{g : P \to Q | g \text{ is affine }\}$,*
$aff(P,Q) := \{g : aff(P) \to aff(Q) | g \text{ is affine }\}$,
and $aff(P) := \{\sum_{i=1}^{n} \lambda_i x_i | n \in \mathbb{N}, x_i \in X \subset V, V \text{ is an } \mathbb{R}- \text{ vector sapce, } \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} = 1\}$.

Proposition 9. $Hom(P,Q) = \cap_{\alpha_j,x_i}\{g \in aff(P,Q)|\phi_{\alpha_j,x_i}(g) \geq 0\}$.

- "$\subseteq$"

  Proof. Since $f \in Hom(P,Q)$, $\alpha_j :$ aff($\mathbb{Q}$) $\to \mathbb{R}$, and $\alpha_j(f(x_i)) \geq 0$, thus $f \in \cap H^+_{\alpha_j}$. □

- "$\supseteq$" consider this direction as a lemma as follows:

  Lemma 2. *Given: $f :$ aff$(P) \to$ aff(Q) such that $\alpha_j(f(x_i)) \geq 0$, $\forall j, i$ where $1 \leq i \leq m$, and $1 \leq j \leq n$.*
  $f(P) \subseteq Q$. That is, to show $\forall x \in P \Rightarrow f(x) \in Q$, where $P = conv(x_1, x_2, ..., x_n)$, and $x = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n$ with $\lambda_i \in \mathbb{R}_+$, $\sum_{i=1}^{n} \lambda_i = 1$.

  Proof. Define the following evaluation map:

$$aff(P,Q) \xrightarrow{\phi_{\alpha_j,x_i}} \mathbb{R}$$

$$g \xmapsto{\phi} \alpha_j(g(x_i))$$

$$f(x) = \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3) + ... + \lambda_n f(x_n)$$

$$\alpha_j(f(x)) = \lambda_1\alpha_j(f(x_1)) + \lambda_2\alpha_j(f(x_2)) + \lambda_3\alpha_j(f(x_3)) + ... + \lambda_n\alpha_j(f(x_n))$$

  where each term $\alpha_j(f(x_i)) \geq 0$ by definition, and $\lambda_i \in \mathbb{R}_+$, and this completes the proof. □

Theorem 22. *$Hom(P,Q) = \cap_{\alpha_j, x_i} H^+_{\phi_{\alpha_i, x_i}} \subset aff(P,Q)$*

Proof. Proof by contradiction. If the claim is not true, and we pick a point z in $\mathrm{Hom}(P,Q)$, then by the definition $\mathrm{Hom}(P,Q) := \{f : P \to Q | f \text{ is affine } \}$, and $aff(P,Q) := \{f : aff(P) \to aff(Q) | \text{ f is affine } \}$. Since $P \subset aff(P)$ and $Q \subset aff(Q)$, hence $z \in aff(P,Q)$. $\square$

Theorem 23. *If $f(P) \subset Q$, then $Hom(P,Q) = \cap_{\alpha_j, x_i} H^+_{\phi_{\alpha_i, x_i}}$ is bounded.*

Proof. Proof by contradiction.
Assume to the contrary, the set is unbounded. WLOG, assume the set is unbounded on positive x_1-axis direction. Then, along the positive x_1-axis, we pick a sequence $\{f_n\}$, $\forall f_i \in Hom(P,Q)$.
Denote the origin as O which is also the zero function of the set that might not be in $Hom(P,Q)$.
Then, $d(f_1, O) =\| f_1 \|= \sup_{t \in P} |f_1(t)|$,
$d(f_1, O) =\| f_1 \|= \sup_{t \in P} |f_1(t)|$,
$d(f_2, O) =\| f_2 \|= \sup_{t \in P} |f_2(t)|$,
$d(f_3, O) =\| f_3 \|= \sup_{t \in P} |f_3(t)|$,
....
$d(f_n, O) =\| f_n \|= \sup_{t \in P} |f_n(t)|$,
....
and so on.
However, we have the given condition $f(P) \subset Q$, since Q is a bounded polytope, so when we take n to infinity, we can have infinitely many f_i with $\sup_{t \in P} |f_i(t)|$ all equal to the maximum of the absolute value of the distance of the point within the set $f(P)$ which is contained the polytope Q to the origin in the ambient space of Q, and let's denote it as α.
Then there exists $N \in \mathbb{N}$, such that $\forall n > N$, we have $\| f_n \|= \alpha$. Thus, α should be the limit point of the sequence (evaluated in P) of (f_n), since $Hom(P,Q)$ is unbounded along this line, then this α should be in the set $Hom(P,Q)$. Likewise, not only in this positive x_1-axis direction, $Hom(P,Q)$ has all of its limit points since Q is bounded, and $f(P) \subset Q$. Then, $Hom(P,Q)$ is closed and bounded (by α which is the largest value of the supremum norm of the function, i.e., the distance between the function f_n to the zero function (which sits in the ambient space) is bounded), and this leads to a contradiction. Therefore, the assumption is false, and $Hom(P,Q)$ is bounded.

$\square$

In the following, let's only consider convex polytopes. Hence, all polytopes are the compact intersections of finitely many affine half-spaces, or equivalently the convex hulls of finitely many points.

For two polytopes in their ambient vector spaces $P \subset V$ and $Q \subset W$, their join is defined by

$$joint(P,Q) = conv[(P,0,0) \cup (0,Q,1)]$$
$$= conv\{(x,0,0),(0,y,1)|x \in P, y \in Q\} \subset V \oplus W \oplus \mathbb{R}$$

where $V \oplus W \oplus \mathbb{R} = V \times W \times \mathbb{R}$.

Let ι_P and ι_Q be the embeddings of P and Q into $join(P,Q)$. Every point $z \in join(P,Q)$ can be uniquely represented as $z = \lambda\iota_P(x) + (1-\lambda)\iota_Q(y)$, $\lambda \in [0,1]$. Then, for two affine maps $g : P \to R$ and $h : Q \to R$ we have the affine map:

$$\phi : join(P,Q) \to R,$$

where

$$\lambda(\iota_P(x)) + (1-\lambda)(\iota_Q(y)) \longmapsto \lambda g(x) + (1-\lambda)h(y),$$

and $\lambda \in [0,1], x \in P, y \in Q$.

Definition 24. *Two polytopes $P \subseteq V$ and $Q \subseteq W$ are affinely isomorphic, written as $P \simeq Q$, if $\exists f : V \longrightarrow W$, where f is affine, and f is a bijective map between the points of the two polytopes.*

Proposition 10. *$Hom(join(P,Q),R) \cong Hom(P,\mathbb{R}) \simeq Hom(P,R) \times Hom(Q,R)$.*

Proposition 11.

$$Hom(P, Q_1 \times Q_2) \simeq Hom(P,Q_1) \times Hom(P,Q_2)$$

Proof.

1. Check: $\alpha : f \mapsto (g,h)$ is surjective
Take $c = (g,h) \in Hom(P,Q_1) \times Hom(P,Q_2)$, then by the definition of tensor product, and the definition of $Hom(P,Q_i)$ that each point in this set is an affine map defined on the domain P, so there exists $t \in P$ as a parameter of each point in $Hom(P,Q_i)$.
Then, for each $t \in P$, we have $g(t) \in Q_1$, $g \in Hom(P,Q_1)$. Similarly, we also have $h(t) \in Q_2$, where $g \in Hom(P,Q_2)$.
Hence, we can put everything together $c(t) = (g,h)(t) = (g(t),h(t))$, where $(g,h) \in Hom(P,Q_1) \times Hom(P,Q_2)$, and $(g(t),h(t)) \in Q_1 \times Q_2$, for any t in P, i.e., for any c evaluated at $t \in P$.

Thus, by using $(g(t), h(t)) \in Q_1 \times Q_2$, since $t \in P$, by using this property, we can find $f \in Hom(P, Q_1 \times Q_2)$ where f is also parametrized by the same single parameter $t \in P$. Since f is a function map to $Q_1 \times Q_2$, and since the input is specified as $t \in P$, hence $f(t) = (g(t), h(t)) \in Q_1 \times Q_2$. Therefore, the map $\alpha : f \mapsto (g, h)$ is surjective.

2. Check: $\alpha : f \mapsto (g, h)$ is injective
If $(g_1(t), h_1(t)) \neq (g_2(t), h_2(t))$, then since $\alpha(f_i(t)) = (g_i(t), h_i(t))$, where $g_i \in Hom(P, Q_1), h_i \in Hom(P, Q_2), f_i \in Hom(P, Q_1 \times Q_2)$, so we have $\alpha(f_1(t)) \neq \alpha(f_2(t))$, and this implies $f_1(t) \neq f_2(t)$. Hence, α is injective.

3. Check: Given f is affine, check g, h, and α are all affine maps
Since P, Q_1, and Q_2 are polytopes, so there are all affine spaces, and for $t \in P$, we can write it in affine combination $t = \sum_{i=1}^{n} a_i x_i$.
Based on the previous two proofs of bijection, we have the bijection between a point f in $Hom(P, Q_1 \times Q_2)$ and $(g, h) \in Hom(P, Q_1) \times Hom(P, Q_2)$, where both are evaluated at $t\ inP$. Hence, again, we can write $f = (g, h)$.
Then, when f is evaluated at t, on the left-hand-side, we have $f(t) = f(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i f(x_i)$. Likewise, on the right-hand-side, $(g, h) = f$, so

$$(g, h)(t) = (g, h)(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i (g, h)(x_i) = \sum_{i=1}^{n} a_i (g(x_i), h(x_i)).$$

Since $(g, h)(t) = (g(t), h(t))$, hence, by the above result, we know $g(t) = \sum_{i=1}^{n} a_i g(x_i)$, and $h(t) = \sum_{i=1}^{n} a_i h(x_i)$, thus g and h are affine. Furthermore, since $\alpha(f(t)) = (g, h)(t)$, and from the above, we have $(g, h)(t) = f(t)$ which is affine, hence $\alpha(f(t))$ is also affine.

4. Check: α is affine isomorphism
Since in "3" we have checked α is affine, and in "1" and "2" we have proved α is a bijection. So we only need to check the following claim:

$$\alpha(f_1 + f_2) = \alpha(f_1) + \alpha(f_2).$$

Since $\alpha((f_1 + f_2)(t)) = ((g_1, h_1) + (g_2, h_2))(t) = (g_1 + g_2, h_1 + h_2)(t) = (g_1(t) + g_2(t), h_1(t) + h_2(t)) = (g_1(t), h_1(t)) + (g_2(t), h_2(t)) = (g_1, h_1)(t) + (g_2, h_2)(t) = \alpha(f_1(t)) + \alpha(f_2(t))$. Therefore, based on all above proofs, α is affine isomorphism.

$\square$

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