$$I = \int d^4x \sqrt{-g}\, R$$

Make variation w.r.t. $g_{\mu\nu}$, keeping the $\underset{①}{\underline{g_{\mu\nu}}}$ and $\underset{②}{\underline{\text{their } 1^{st} \text{ derivatives}}}$ constant on the bamdary.

Claim:

Putting $\delta I = 0$ for arbitrary $\delta g_{\mu\nu}$ gives Einstein's vacuum equations.

(i) $R_{\mu\nu} = \partial_\nu \Gamma^\alpha_{\mu\alpha} - \partial_\alpha \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}$

$$\Rightarrow R = g^{\mu\nu} R_{\mu\nu}$$

$$= \underbrace{g^{\mu\nu}(\partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu})}_{R^*} - \underbrace{g^{\mu\nu}(\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho})}_{L}$$

$$= R^* - L$$

involving 2^{nd} derivatives

But, they occur only linearly, so they can be removed by partial integration.

$$\Rightarrow R^* \sqrt{-g} = \underbrace{\partial_\nu (g^{\mu\nu}\Gamma^\sigma_{\mu\sigma}\sqrt{-g}) - \partial_\sigma (g^{\mu\nu}\Gamma^\sigma_{\mu\nu}\sqrt{-g})}_{\text{Total differential : contribute nothing to } I}$$
$$- \partial_\nu (g^{\mu\nu}\sqrt{-g})\Gamma^\sigma_{\mu\sigma} + \partial_\sigma (g^{\mu\nu}\sqrt{-g})\Gamma^\sigma_{\mu\nu}$$

$$\left[\begin{aligned} &\because\ \partial_\nu (g^{\mu\nu}\sqrt{-g}) = -g^{\nu\beta}\Gamma^\mu_{\beta\nu}\sqrt{-g} \\ &\text{and } \partial_\sigma (g^{\mu\nu}\sqrt{-g}) = (-g^{\nu\beta}\Gamma^\mu_{\beta\sigma} - g^{\mu\alpha}\Gamma^\nu_{\alpha\sigma} + g^{\mu\nu}\Gamma^\beta_{\sigma\beta})\sqrt{-g} \end{aligned}\right]$$

$$\Rightarrow \int d^4x\, R^* \sqrt{-g} = \int d^4x \left(g^{\nu\beta}\Gamma^\mu_{\beta\nu}\Gamma^\sigma_{\mu\sigma}\sqrt{-g} - 2g^{\nu\beta}\Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\mu\nu}\sqrt{-g} + g^{\mu\nu}\Gamma^\beta_{\sigma\beta}\Gamma^\sigma_{\mu\nu}\sqrt{-g} \right)$$

$$\Rightarrow \int d^4x R^* \sqrt{-g} = \int -2L\sqrt{-g}\, d^4x$$

$$\Rightarrow I = \int L\sqrt{-g}\, d^4x$$

which involves only the $g_{\mu\nu}$ and their first derivatives.

Put $\mathcal{L} = L\sqrt{-g}$.

We take it as the action density for the gravitational field.
It's not a scalar density.
But it's more ~~conve~~ convenient than $R\sqrt{-g}$, which is a scalar density, because it does not involve 2^{nd} derivatives of the $g_{\mu\nu}$.

According to the ordinary ideas of dynamics, the action is the time ~~interval~~ integral of the Lagrangian.

We have

$$I = \int \mathcal{L}\, d^4x$$
$$= \int x_0 \int \mathcal{L}\, dx^1 dx^2 dx^3$$

So the Lagrangian is evidently

$$\int \mathcal{L}\, dx^1 dx^2 dx^3.$$

Thus $\mathcal{L}$ may be considered as Lagrangian density (in 3D) as well as the action density (in 4D).
We may look upon the $g_{\mu\nu}$ as dynamical coordinates and their time derivatives as the velocities.

Then the Lagrangian is quadratic (nonhomogeneous) in the velocities, as it usually is in ordinary dynamics.

Vary $\mathcal{L}$. with using $\Gamma^{\mu}_{\nu\mu}\sqrt{-g} = \partial_\nu \sqrt{-g}$

$$\partial_\nu\sqrt{-g} = \frac{-1}{2}\frac{1}{\sqrt{-g}}\partial_\nu g = \frac{+1}{2}\frac{\sqrt{-g}}{g}\partial_\nu g$$

$\because$ (i) $\partial_\nu g = \frac{2g}{\sqrt{-g}}\partial_\nu\sqrt{-g}$.

and $\partial_\nu g = g g^{\lambda\mu}\partial_\nu g_{\lambda\mu}$

$$\Rightarrow \cancel{g} g^{\lambda\mu}\partial_\nu(g_{\lambda\mu})\sqrt{-g} = 2\cancel{g}\,\partial_\nu\sqrt{-g}$$

$$\Rightarrow \boxed{\partial_\nu\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\lambda\mu}\partial_\nu g_{\lambda\mu}}$$

(ii) $\Gamma^{\mu}_{\nu\mu} = g^{\lambda\mu}\Gamma_{\lambda\nu\mu} = \frac{1}{2}g^{\lambda\mu}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$

$$= \frac{1}{2}g^{\lambda\mu}\partial_\nu(g_{\lambda\mu}).$$

$$\Rightarrow \Gamma^{\mu}_{\nu\mu}\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\lambda\mu}\partial_\nu g_{\lambda\mu} = \partial_\nu\sqrt{-g}$$

$$\delta\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}\, g^{\mu\nu}\sqrt{-g}\right)$$

$$= \Gamma^{\alpha}_{\mu\nu}\,\delta\left(\Gamma^{\beta}_{\alpha\beta}\, g^{\mu\nu}\sqrt{-g}\right) + \Gamma^{\beta}_{\alpha\beta}\, g^{\mu\nu}\sqrt{-g}\;\delta\Gamma^{\alpha}_{\mu\nu}$$

$$= \Gamma^{\alpha}_{\mu\nu}\,\delta\left(g^{\mu\nu}\,\partial_\alpha\sqrt{-g}\right) + \Gamma^{\beta}_{\alpha\beta}\,\delta\left(\Gamma^{\alpha}_{\mu\nu}\, g^{\mu\nu}\sqrt{g}\right)$$
$$- \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{-g}\right)$$

$$= \Gamma^{\alpha}_{\mu\nu}\,\delta\left(g^{\mu\nu}\,\partial_\alpha\sqrt{-g}\right) - \Gamma^{\beta}_{\alpha\beta}\,\delta\left(\partial_\nu\left(g^{\alpha\nu}\sqrt{g}\right)\right)$$
$$- \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{-g}\right)$$

$$\delta\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\, g^{\mu\nu}\sqrt{-g}\right)$$

$$= 2\left(\delta\Gamma^{\beta}_{\mu\alpha}\right)\Gamma^{\alpha}_{\nu\beta}\, g^{\mu\nu}\sqrt{-g} + \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\,\delta\left(g^{\mu\nu}\sqrt{g}\right)$$

$$= 2\,\delta\left(\Gamma^{\beta}_{\mu\alpha}\, g^{\mu\nu}\sqrt{-g}\right)\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\,\delta\left(g^{\mu\nu}\sqrt{g}\right)$$

$$= -\,\delta\left(g^{\nu\beta}\,\partial_\alpha\sqrt{-g}\right)\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\,\delta\left(g^{\mu\nu}\sqrt{g}\right)$$

$$\Rightarrow \delta\mathcal{L} = \Gamma^{\alpha}_{\mu\nu}\,\delta\partial_\alpha\left(g^{\mu\nu}\sqrt{-g}\right) - \Gamma^{\beta}_{\alpha\beta}\,\delta\partial_\nu\left(g^{\alpha\nu}\sqrt{g}\right)$$
$$+ \left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\right)\delta\left(g^{\mu\nu}\sqrt{g}\right)$$

$$= \cancel{\partial_\alpha\left(\Gamma^{\alpha}_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{g}\right)\right)} - \partial_\alpha\Gamma^{\alpha}_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{g}\right)$$
$$- \cancel{\partial_\nu\left(\Gamma^{\beta}_{\alpha\beta}\,\delta\left(g^{\alpha\nu}\sqrt{g}\right)\right)} + \partial_\nu\Gamma^{\beta}_{\alpha\beta}\,\delta\left(g^{\alpha\nu}\sqrt{g}\right)$$
$$+ \left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta} - \Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\mu\nu}\right)\delta\left(g^{\mu\nu}\sqrt{g}\right)$$

$$\Rightarrow \delta I = \delta\int \mathcal{L}\, d^4x = \int R_{\mu\nu}\,\delta\left(g^{\mu\nu}\sqrt{g}\right) d^4x.$$

with the $\delta g_{\mu\nu}$ arbitrary, the quantities $\delta(g^{\mu\nu}\sqrt{g})$ are also independant and arbitrary.

$$\boxed{\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}}$$

$$\because (\partial_\sigma g^{\alpha\mu}) g_{\mu\nu} + g^{\alpha\mu}(\partial_\sigma g_{\mu\nu})$$

$$= \partial_\sigma (g^{\alpha\mu} g_{\mu\nu})$$

$$= \partial_\sigma (g^{\alpha}_{\nu})$$

$$= 0$$

$$\Rightarrow g_{\mu\nu}(\partial_\sigma g^{\alpha\mu}) = -g^{\alpha\mu}(\partial_\sigma g_{\mu\nu})$$

$$\Rightarrow g^{\beta\nu} g_{\mu\nu}(\partial_\sigma g^{\alpha\mu}) = -g^{\beta\nu} g^{\alpha\mu}(\partial_\sigma g_{\mu\nu})$$

$$\Rightarrow \partial_\sigma g^{\alpha\beta} = -g^{\beta\nu} g^{\alpha\mu}(\partial_\sigma g_{\mu\nu})$$

$$\Rightarrow \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$$

$$\boxed{\delta\sqrt{-g} = \frac{1}{2} g^{\alpha\beta} \delta g_{\alpha\beta}}$$

$$\Rightarrow \delta(g^{\mu\nu}\sqrt{-g}) = -(g^{\mu\alpha} g^{\nu\beta} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta})\sqrt{-g}\,\delta g_{\alpha\beta}$$

$$\Rightarrow \delta I = -\int R_{\mu\nu}(g^{\mu\alpha} g^{\nu\beta} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta})\sqrt{-g}\,\delta g_{\alpha\beta}\, d^4x$$

$$= -\int (R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R)\sqrt{-g}\,\delta g_{\alpha\beta}\, d^4x$$

$$= 0$$

$$\Rightarrow R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R = 0$$

(Einstein's Vaccuum Eq.)

The usual expression for the action density of the electromagnetic field is $\frac{1}{8\pi}(E^2 - H^2)$

If we write it in the 4-D notation of special relativity given in the form

$$\frac{-1}{16\pi} F_{\mu\nu} F^{\mu\nu}.$$

This leads to the expression

$$I_{em} = \frac{-1}{16\pi} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d^4x$$

For the invariant action in general relativity.

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

$$R = \Omega^2 (\tilde{R} + 6 \tilde{\Box} \omega - 6 \tilde{g}^{\mu\nu} \partial_\mu \omega \partial_\nu \omega)$$

$$\omega \equiv \ln \Omega$$

$$\partial_\mu \omega \equiv \frac{\partial \omega}{\partial \tilde{x}^\mu}$$

$$\tilde{\Box} \omega \equiv \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left(\sqrt{-\tilde{g}}\, \tilde{g}^{\mu\nu} \partial_\nu \omega \right)$$

$$\Omega^{-2} = F = \exp\left[\sqrt{\tfrac{2}{3}} \kappa \phi\right]$$

$$\Omega^{-4} = F^{-2} = \frac{1}{e^{2\sqrt{\frac{2}{3}}\kappa\phi}} = e^{-2\sqrt{\frac{2}{3}}\kappa\phi}$$

$$\Omega^{-8} = e^{-4\sqrt{\frac{2}{3}}\kappa\phi}$$

$$S = \underset{\uparrow\, K=\sqrt{8\pi G}}{\frac{1}{2K^2}}\int d^4x\sqrt{-g}\, f(R) + \int d^4x\sqrt{-g}\left(\frac{-1}{16\pi}F_{\mu\nu}F^{\mu\nu}\right) + \int d^4x\sqrt{-g}\,\mathcal{L}_M(g_{\mu\nu}, \Psi_M)$$

$$= \int d^4x\sqrt{-g}\left(\frac{1}{2K^2}FR - \underbrace{\frac{FR-f}{2K^2}}_{= U}\right) + \int d^4x\sqrt{-g}\left(\frac{-1}{16\pi}F_{\mu\nu}F^{\mu\nu}\right) + \int d^4x\sqrt{-g}\,\mathcal{L}_M(g_{\mu\nu}, \Psi_M)$$

$$= \int d^4x\sqrt{-g}\left(\frac{1}{2K^2}FR - U\right) + \int d^4x\sqrt{-g}\left(\frac{-1}{16}g_{\alpha\mu}g_{\beta\nu}F^{\alpha\beta}F^{\mu\nu}\right) + \int d^4x\sqrt{-g}\,\mathcal{L}_M(g_{\mu\nu}, \Psi_M)$$

$$= \int d^4x\sqrt{-\tilde{g}}\,\Omega^{-4}\left(\frac{1}{2K^2}F\left(\Omega^2\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega\right) - U\right)$$

$$+ \int d^4x\sqrt{-\tilde{g}}\,\Omega^{-8}\left(\frac{-1}{16}\tilde{g}_{\alpha\mu}\tilde{g}_{\beta\nu}F^{\alpha\beta}F^{\mu\nu}\right)$$

$$+ \int d^4x\sqrt{-\tilde{g}}\,\Omega^{-4}\mathcal{L}_M\left(\Omega^{-2}\tilde{g}_{\mu\nu}, \Psi_M\right)$$

$$\Omega^2 = F$$

$$K\phi = \sqrt{\frac{3}{2}}\ln F \qquad \ln\Omega^2 = \ln F$$

$$\ln\Omega = \frac{1}{2}\ln F$$

$$\ln F = \sqrt{\frac{2}{3}}K\phi$$

$$= \frac{1}{2}\left(\sqrt{\frac{2}{3}}K\phi\right) = \frac{1}{\sqrt{6}}K\phi = \omega$$

$$= \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2K^2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

$$- \frac{1}{16}\int d^4x\sqrt{-\tilde{g}}\left[F^{-4}(\phi)\,\tilde{g}_{\alpha\mu}\tilde{g}_{\beta\nu}F^{\alpha\beta}F^{\mu\nu}\right]$$

$$+ \int d^4x\sqrt{-\tilde{g}}\left[F^{-2}(\phi)\,\mathcal{L}_M\left(F^{-1}(\phi)\tilde{g}_{\mu\nu}, \Psi_M\right)\right]$$