§ The general features of $f(R)$ theory §

Let $S = \int d^4x \sqrt{-g}\, f(R) + \mathcal{L}_m$ be the gravitational action where $f(R)$ is the generic function of the Ricci scalar R.

Note:
GR is recovered in the particular case $f(R) = \frac{-R}{16\pi G}$, and $\mathcal{L}_m$ is the action for a perfect fluid minimally coupled with gravity.

$f(R)$ - Formalism :

Metric formalism.

Palatini formalism.

Metric-affine formalism

Metric Formalism.

The action leads to 4th order differential equations

$$f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu} \Box f_R = \frac{-1}{2} T^m_{\mu\nu}$$

where a subscript R denotes differentiation with respect to R and $T^m_{\mu\nu}$ is the matter fluid stress-energy tensor.

Question : I want to derive the "cosmological equations" in a Friedmann-Robertson-Walker (FRW) metric.

Step 1. **Define :**

$\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R})$, $Q = \{a, R\}$

$TQ = \{a, \dot{a}, \dot{R}, R\}$

where $\mathcal{L}$ is a canonical Lagrangian, Q is the configuration space. TQ is the related tangent bundle on which $\mathcal{L}$ is defined.

$$\left.\begin{array}{l} a = a(t) \text{ is scale factor} \\ R(t) \text{ is Ricci scalar.} \end{array}\right\} \text{in FRW metric.}$$

Step 2. Use the method of the Lagrange multipliers to set R as a constraint of the dynamics.

Selecting the suitable Lagrange multiplier and integrating by parts, the Lagrangian $\mathcal{L}$ becomes canonical.

In our case, we have

$$S = 2\pi^2 \int dt\, a^3 \left\{ f(R) - \lambda \left[R + 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}$$

Note: It's straightforward to show that, for $f(R) = -R/16\pi G$, one obtains the usual Friedman equations.

Step 3. The variation with respect to R of the Lagrange multiplier gives $\lambda = f_R$.

Therefore, integrating by parts, the point-like FLRW Lagrangian is

$$\mathcal{L} = a^3(f - f_R R) + 6a^2 f_{RR} \dot{R} \dot{a}$$

$$+ 6 f_R a \dot{a}^2 - 6k f_R a$$

which is a canonical function of two coupled fields, R and a, both depending on time, t.

Step 4. The total energy $E_{\mathcal{L}}$, corresponding to the $\{0,0\}$ - Einstein equation, is

Total energy
$$E_{\mathcal{L}} = 6 f_{RR}\, a^2 \dot{a} \dot{R} + 6 f_R \dot{a} a^2 - a^3 (f - f_R R) + 6 k f_R a = D$$

where D represents the standard amount of dust fluid as, for example, measured today.

E.O.M. The equations of motion for $a(t)$ and $R(t)$ are respectively

$$\begin{cases} f_{RR}\left[R + 6H^2 + 6\dfrac{\ddot{a}}{a} + 6\dfrac{k}{a^2} \right] = 0 \\ 6 f_{RRR} \dot{R}^2 + 6 f_{RR} \ddot{R} + 6 f_R H^2 + 12 f_R \dfrac{\ddot{a}}{a} = 3(f - f_R R) - 12 f_{RR} H \dot{R} - 6 f_R \dfrac{k}{a^2} \end{cases}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter.

Question : (1) What's the form of the function $f(R)$?

(2) What's the solution of the system

(i) Totol energy eq.

(ii) two of e.o.m.

Hint:

(*) which can be achieved by asking for the existence of Noether symmetries.

Note:

(*) Noether symmetries guarantees
=> the reduction of dynamics and the eventually solvability system.

Analysis of solutions

Property must have

1) FACT:

We need a cosmological solution of the field equations which exhibits not only an (1) accelerated phase 展出 in recent universe, but also (2) a decelerated period, which lasts for a long time, sufficient to allow the formation of structures.

2) By Noether symmetry approach, ~~obtaing~~ could obtain a general exact solution of the equations

Noether Symmetry Approach.

1. Assumption. Suppose $\exists$ a vector field X, where

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{R}}$$

s.t. the Lie derivative of the Lagrangian is zero, i.e. $\mathcal{L}$ is conserved and X is a Noether symmetry.

2. $\alpha = ?$ $\beta = ?$ Thus, it's possible to find

$$\alpha = \frac{1}{a} \; ; \; \beta = \frac{-2R}{a^2} \; ; \; f(R) = -|R|^{\frac{3}{2}}.$$

[Note that : the absolute value is needed, since here $R < 0$.
we let. (by conventions)]

After obtain the symmetry, we have an additional constant of the motion, and it is then easy to find a change of variable

$$\{a, R\} \longrightarrow \{u, v\}, \text{ such that}$$

one of the variable is cyclic.

3. In our case, $\Rightarrow \begin{cases} u = a^2 |R| \\ v = a^2/2 \end{cases} \Rightarrow$ The new Lagrangian is $\mathcal{L}' = \frac{u^{3/2}}{2} + \frac{9}{2} \frac{\dot{u}\dot{v}}{\sqrt{u}} - 9k\sqrt{u}$

4. Noether Charge

The Noether charge is then

$$\Sigma_1 = \dot{u}/\sqrt{u},$$

leading to immediate integration for u.

5.

Introducing the solution into

$$E_{\mathcal{L}} = D,$$

and solving for v, we obtain.

$$u = \frac{1}{4}(\Sigma_1 t + \Sigma_0)^2$$

$$v = \frac{\Sigma_1^2}{288}t^4 + \frac{\Sigma_1 \Sigma_0}{72}t^3$$

$$+ \left(\frac{\Sigma_0^2}{48} - \frac{k}{2}\right)t^2$$

$$+ \left(\frac{\Sigma_0^3}{72\Sigma_1} - k\frac{\Sigma_0}{\Sigma_1} + \frac{2D}{9\Sigma_1}\right)t$$

$$+ V_0.$$

where the parameters

Σ_0, Σ_1, D and V_0 are the integration constants of the equation.

They are four since this is a general solution of a fourth order problem.

6. $a(t) = ?$ Coming back to $a(t)$, and setting, for the sake of simplicity $\dot{a}(0) = 0$ i.e. $V_0 = 0$

we obtain $a = \sqrt{a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t}$

with $a_4 = \dfrac{\Sigma_1^2}{144}$;

$$a_3 = \frac{\Sigma_1 \Sigma_0}{36};$$

$$a_2 = \frac{\Sigma_0^2}{24} - k;$$

$$a_1 = \frac{\Sigma_0^3}{36 \Sigma_1} - 2k\frac{\Sigma_0}{\Sigma_1} + \frac{4D}{9\Sigma_1}$$

$$a(t) = \begin{cases} \propto t^2 & \text{for large } t \\ \propto t^{1/2} & \text{for small } t. \end{cases}$$

$$S=\int d^4x\sqrt{-g}\,f(R)$$

FLRW $\rightarrow$ $$=2\pi^2\int a^3 f(R)\,dt.$$

↓ Lagrange multiplier.

$$S=2\pi^2\int dt\left\{f(R)a^3-\lambda(R-R)\right\}$$

$$=2\pi^2\int dt\left\{f(R)a^3-\lambda\left[R+6\left(\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}+\frac{k}{a^2}\right)\right]\right\}$$

↓ inorder to determine λ.
very with respect to R.

$$a^3\frac{df}{dR}\delta R-\lambda\delta R=0$$

$$\Rightarrow \lambda=a^3 f'(R).$$

$$\Rightarrow S=2\pi^2\int dt\left\{f(R)a^3-a^3f'(R)R\right.$$
$$\left.-a^3f'(R)\,6\left(\frac{\ddot a}{a}+\frac{\dot a^2}{a^2}+\frac{k}{a^2}\right)\right\}$$

After integration by part.

$$\mathcal{L}=a^3[f(R)-Rf'(R)]+6\dot a^2 a f'(R)+6a^2\dot a\dot R f''(R)$$
$$-akf'(R)$$

$$\mathcal{L} = \mathcal{L}(a, \dot{a}, R, \dot{R})$$

$$\frac{\partial L}{\partial a} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{a}}\right) = 0$$

$$\frac{\partial L}{\partial a} = 3a^2[f(R) - Rf'(R)] + 6\dot{a}^2 f'(R) + 12a\dot{a}\dot{R}f''(R)$$
$$- kf'(R).$$

$$\frac{\partial L}{\partial \dot{a}} = 12\dot{a}af'(R) + 6a^2\dot{R}f''(R)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{a}}\right) = 12\ddot{a}af' + 12\dot{a}^2 f' + 12a\dot{a}\dot{R}f''$$
$$+ 6a^2\ddot{R}f'' + 6a^2\dot{R}^2 f'''(R)$$

$$\frac{\partial L}{\partial a} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{a}}\right) = 0 \ ; \ \frac{\partial L}{\partial R} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{R}}\right) = 0.$$

$$\left(\frac{\ddot{a}}{a}\right)f'(R) + 2\left(\frac{\dot{a}}{a}\right)f''(R)\dot{R} + f''(R)\ddot{R} + f'''(R)\dot{R}^2$$
$$- \frac{1}{2}[Rf'(R) + f(R)] = 0$$

$$6\dot{a}^2 af'(R) - a^3[f(R) - Rf'(R)] + 6a^2\dot{a}\dot{R}f''(R)$$
$$+ akf'(R) = 0.$$

$$R = -6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)$$

Thy symmetry generator is defined on the tangent bundle $TQ(a, \dot{a}, R, \dot{R})$ and it is

$$X = \alpha(a, R)\frac{\partial}{\partial a} + \beta(a, R)\frac{\partial}{\partial R} + \frac{d\alpha}{dt}\frac{\partial}{\partial \dot{a}} + \frac{d\beta}{dt}\frac{\partial}{\partial \dot{R}}$$

while the Nöther condition $L_X \mathcal{L} = 0$ produces the system

$$f'(R)\left[\alpha + 2a\frac{\partial \alpha}{\partial a}\right] + af''(R)\left[\beta + a\frac{\partial \beta}{\partial a}\right] = 0$$

$$a^2 f''(R)\frac{\partial \alpha}{\partial R} = 0 \rightarrow \alpha \text{ is a func. of } a \text{ only. if } f'' \neq 0$$

$$2f'(R)\frac{\partial \alpha}{\partial R} + f''(R)\left[2\alpha + a\frac{\partial \alpha}{\partial a} + a\frac{\partial \beta}{\partial R}\right] + \alpha\beta f'''(R) = 0$$

$$3\alpha\left[f(R) - Rf'(R)\right] - \alpha\beta R f''(R) = 0$$

$$\alpha f'(R) + \alpha\beta f''(R) = 0$$

The symmetry is given by the functions

$$\alpha = \frac{\beta_0}{a}, \quad \beta = -2\beta_0\frac{R}{a^2}, \quad f(R) = f_0 R^{3/2}$$

which solve the above system.

β_0, f_0 are integration constants.

The new variables induced by

$$\mathcal{L}(a, \dot{a}, \varphi, \dot{\varphi}) \to \tilde{\mathcal{L}}(w, \dot{w}, \dot{z})$$

$$i_X dz = \alpha \frac{\partial z}{\partial a} + \beta \frac{\partial z}{\partial \varphi} = 1$$

$$i_X dw = \alpha \frac{\partial w}{\partial a} + \beta \frac{\partial w}{\partial a} = 0$$

~~From~~

$$\cancel{a^2 f'' \frac{\partial \alpha}{\partial R} = 0}$$

~~we have that α is a func. of a only, if we want to avoid trivial~~

from which the $\mathcal{L}$ becomes

$$\tilde{\mathcal{L}} = \frac{9}{2} \beta_0 \frac{\dot{z}\dot{w}}{\sqrt{w}} - 9k\sqrt{w} - \frac{1}{2}\sqrt{w^3}$$

which can be rewritten in the form

$$\tilde{\mathcal{L}} = 9\beta_0 \dot{z}\dot{y} - 9ky - \frac{1}{2}y^3.$$

using $y = \sqrt{w}$.

The dynamics is then described from the equations

$$\ddot{y} = 0, \text{ from which}$$

$$\dot{y} = \dot{y}_0 = \Sigma_0$$

$$9\beta_0 \ddot{z} + 9k + \frac{3}{2} y^2 = 0$$

$$9\beta_0 \dot{y}\dot{z} + 9ky + \frac{1}{2} y^3 = 0$$

$$y(t) = \dot{y}_0 t + y_0,$$

$$z(t) = C_4 t^4 + C_3 t^3 + C_2 t^2 + C_1 t + C_0$$

$$C_4 = \frac{-\dot{y}_0^2}{72\beta_0}$$

$$C_3 = \frac{-\dot{y}_0 y_0}{3\beta_0}$$

$$C_2 = -\frac{y_0^2}{12\beta_0} - \frac{k}{2}.$$

$$C_1 = \dot{z}_0$$

$$C_0 = z_0$$

The energy condition gives the relation among the initial data. Going back to the physical variables, we have

$$a(t) = \pm\sqrt{d_4 t^4 + d_3 t^3 + d_2 t^2 + d_1 t + d_0}$$

where the d_i's are c_i's multiplied by $2\beta_0$ (we can neglect the minus sign if we consider only expansion) and

$$R = \frac{(\dot{y}_0 t + y_0)^2}{d_4 t^4 + d_3 t^3 + d_2 t^2 + d_1 t^1 + d_0}$$

we have to note that $a(t)$ is a bounded function, since d_4, the coefficient of the leading term, is always negative.

If $\dot{y}_0 = 0$, we have $y_0 = 0$ or

$y_0^2 = -3k$ and then

$$a(t) = \sqrt{c_1 t + c_0}$$

or

$$a(t) = \sqrt{c_2 t^2 + c_1 t + c_0}$$

which are unbounded.