# Lecture 8: Using GAP for Modular Character Theory

**Goal:** Learn how to use GAP to compute ordinary and Brauer character tables, identify $p$-regular classes, determine decomposition matrices, and study blocks in finite groups such as $A_5$, $S_4$, and $S_5$.

## 1. Setup and Character Tables in GAP

To begin, open GAP and load the group whose character theory you want to study.

**Example 8.1 (Loading $A_5$ and viewing its character table):**

```
gap> G := AlternatingGroup(5);
gap> tbl := CharacterTable(G);
gap> Display(tbl);
```

This prints the ordinary complex character table of $A_5$, listing irreducible characters $\chi_1, \chi_2, \ldots$, their degrees, and values on each conjugacy class.

## 2. Viewing $p$-Regular Classes and Modular Data

In modular character theory, only $p$-regular classes (elements whose order is not divisible by $p$) matter.

**Example 8.2 (Show 2-regular classes):**

```
gap> p := 2;
gap> modtbl := DecompositionMatrix(tbl, p);
gap> modtbl;
gap> PositionsProperty(tbl.orders, x -> Gcd(x, p) = 1);
```

This lists the positions of conjugacy classes in $G$ whose elements are $p$-regular.

## 3. Computing Brauer Characters (modular irreducibles)

To obtain the Brauer character table modulo a prime $p$:

```
gap> brauer := BrauerTable(tbl, p);
gap> Display(brauer);
```

**Example 8.3 (Brauer characters of $S_4$ mod 2):**

```
gap> S4 := SymmetricGroup(4);
gap> t := CharacterTable(S4);
gap> Display(t);
gap> b := BrauerTable(t, 2);
gap> Display(b);
```

## 4. Viewing the Decomposition Matrix

**Example 8.4 (Decomposition matrix for $S_4$, mod 2):**

```
gap> dec := DecompositionMatrix(t, 2);
gap> dec;
```

The rows correspond to ordinary irreducibles $\chi_i$, the columns to Brauer irreducibles $\varphi_j$, and entries $d_{ij}$ give the multiplicities:

$$\chi_i|_{p\text{-reg}} = \sum_j d_{ij}\varphi_j$$

# 5. Block Structure in GAP

**Example 8.5 (Block information):**

```
gap> Blocks(tbl, 2);
gap> BlockIdempotents(tbl, 2);
```

These commands provide the block structure of the group algebra $\mathbb{F}_2[G]$ and display block idempotents when available.

# 6. Suggested Explorations

- Try running the same computations for $S_5$, $A_6$, and $\mathrm{PSL}(2,7)$.
- Investigate other primes (e.g., $p = 3, 5$) to see changes in Brauer tables and decomposition matrices.
- Compare the rank of the decomposition matrix with the number of $p$-regular classes.

# 7. Limitations

- GAP does not always compute Brauer tables from scratch; many are precomputed in the `ctbllib` package.
- Some groups may not have modular data stored for all primes.
- To access the full library, use `LoadPackage("ctbllib")`.

# 8. Summary

In this computational lecture, we learned:

- How to compute ordinary and Brauer character tables in GAP,
- How to extract decomposition matrices,
- How to identify $p$-regular classes and block decompositions,
- How to connect theoretical character theory with computational tools.

**Coming Up in Lecture 9:** We'll explore *Green correspondences*, understanding how projective and simple modules relate across subgroup chains in modular representation theory.