Lecture 10: Synthesis, Advanced Examples, and Research Directions in Modular Character Theory

Goal: Connect all prior topics in modular representation theory; explore key examples and global patterns in decomposition matrices and blocks; and outline current research directions, including derived categories, local-global conjectures, and computational challenges.

1. Modular Theory Recap and Concept Map

Core Components of Modular Representation Theory:

- Ordinary Representations: $\chi \in \mathrm{Irr}(G)$
- Modular Representations: $\varphi \in \mathrm{IBr}(G)$
- Decomposition Matrix: $\chi \mapsto \sum d_{\chi\varphi}\varphi$
- Blocks: Partition of irreducibles via primitive idempotents
- Defect Groups: Measure the complexity of blocks
- Projective Indecomposables: Correspond to columns of the decomposition matrix
- Brauer Characters: Class functions on p-regular elements
- Green Correspondence: Transfers indecomposable modules across subgroup chains

Concept Flow Diagram:

$$\mathrm{Irr}(G) \longrightarrow \text{Decomposition Matrix} \longrightarrow \mathrm{IBr}(G) \longrightarrow \text{Blocks} \longrightarrow \text{Defect Groups} \longrightarrow \text{Projectives}$$

2. Advanced Examples and Patterns

Example 10.1 (Alternating Group A_5):

- $|A_5| = 60$
- Ordinary irreducibles: degrees 1, 3, 3, 4, 5
- Over $\mathbb{F}_2$: 3 irreducible Brauer characters
- Blocks: Principal block has nontrivial defect; one block of defect zero

Decomposition Matrix:

χ	φ_1	φ_2	φ_3
χ_1	1	0	0
χ_2	1	1	0
χ_3	1	0	1
χ_4	0	1	1
χ_5	0	0	1

This structure reflects the internal symmetry and modular block decomposition of A_5 over $\mathbb{F}_2$.

3. Deep Theorems and Open Conjectures

Alperin's Weight Conjecture: The number of simple modules in a block equals the number of conjugacy classes of weights (defect group + character pair).

Brauer's $k(B)$-Conjecture: The number of irreducible characters in block B is $\leq |D|$, where D is a defect group of B.

Donovan's Conjecture: For a fixed defect group, only finitely many Morita equivalence classes of blocks exist.

Broué's Abelian Defect Group Conjecture: If B has abelian defect group, then it is derived equivalent to its Brauer correspondent.

4. Computational Research Topics

- Classification of blocks in quasi-simple or Lie-type groups
- Automating Green correspondents and vertex detection
- Computing derived equivalences via Rickard complexes
- Extending Brauer tables using GAP + CTblLib

Tools: GAP, Magma, SageMath, Atlas of Finite Group Representations.

5. Higher-Categorical and Geometric Approaches

Derived Categories: Extend classical module theory to homological settings; two blocks are derived equivalent if their derived categories are equivalent.

Deligne–Lusztig Theory: Describes representations of finite groups of Lie type using étale cohomology of algebraic varieties.

Modular Tensor Categories: Appear in TQFT and CFT, where modular representations give rise to fusion rules and braided structures.

6. Pitfalls and Warnings

- Brauer characters are undefined on p-singular elements
- Induction and restriction may not preserve irreducibility over $\mathbb{F}_p$
- Decomposition matrices are not invertible in general
- Projective modules are not simple unless the algebra is semisimple

7. Final Summary

In this series, we have:

- Developed a solid foundation in modular representation theory,
- Explored the structure of blocks, decomposition matrices, and projective modules,
- Used computational tools (GAP) to study Brauer characters and Green correspondence,
- Connected classical and modern research directions, from conjectures to categorification.

You now have the toolkit to pursue research, computation, or advanced study in modular representation theory.

May your mathematical journey continue with symmetry, structure, and insight.