Two Approaches to Understand Gravitational Lensing

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1 Question

Prove that the size of the Einstein ring, i.e., the Einstein radius, is (in radius)

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_L D_S}}. \tag{1}$$

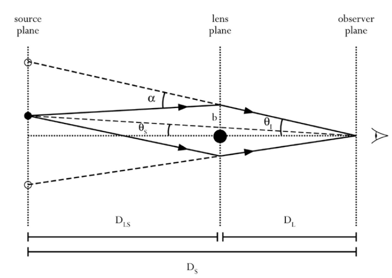

Figure 1: In general, over cosmological distance $D_{LS} \neq D_S - D_L$.

2 Solution

2.1 Newtonian Derivation Is Juxtaposed With Einstein's Derivation

2.1.1 Newtonian Derivation

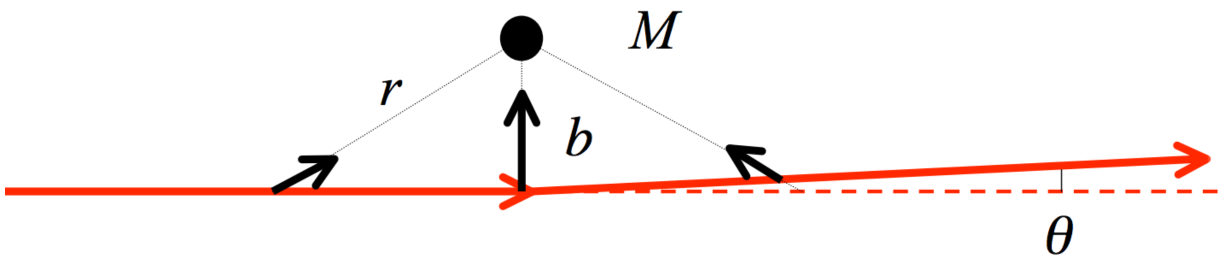

Figure 2: Newtonian Bend Lensing

M denotes point mass, b is impact parameter, θ is the bending angle. Consider the vertical acceleration:

$$g_y = \left(\frac{GM}{r^2}\right)\left(\frac{b}{r}\right) \leq \frac{GM}{b^2} = g_{max}. \tag{2}$$

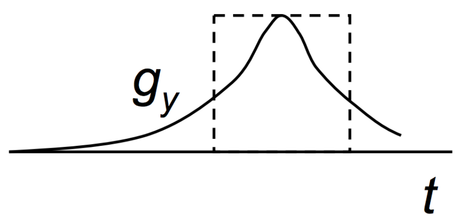

Figure 3: Vertically gravitational strength versus time.

According to equation (2), Figure 2 and Figure 3, we can calculate the average time as a photon(or a light pulse, a quanta) go by the point mass, M.

$$\Delta t \approx \frac{2b}{V_x} \tag{3}$$

Then the vertical velocity can be calculated as follows,

$$V_y = \int g_y dt \tag{4}$$
$$\approx g_{max.}\Delta t \tag{5}$$
$$\approx \left(\frac{GM}{b^2}\right)\left(\frac{2b}{V_x}\right) \tag{6}$$
$$= \frac{2GM}{bV_x}\ . \tag{7}$$

Hence the bend angle:

$$\theta_N \approx \frac{V_y}{V_x} \tag{8}$$
$$\approx \frac{2GM}{bV_x^2} \tag{9}$$
$$\approx \frac{2GM}{bc^2}\ , \tag{10}$$

for photon, $V_x = c$.

2.1.2 Einstein's Derivation

Set Newton constant G=1. In the Einstein's equation, we have

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{11}$$

On the right hand side, contains all possible energy and matter, include light. The left hand side, stands for geometry of the spacetime, is determined by the right. This means that light gives rise to a gravitational field. In the same manner, light is also affected by a gravitational field. Now consider in a Schwarzschild spacetime. By using the expression for relativistic energy:

$$\frac{d\tau}{dt} = \frac{1 - \frac{2M}{r}}{E/m} \tag{12}$$

where τ is the proper time. For total energy per mass as

$$\Delta t = \frac{E/m}{1 - \frac{2M}{r}}\tau. \tag{13}$$

Similarly we can use that the angular momentum per massL/m is a constant of motion

$$\frac{L}{m} = r^2\frac{d\phi}{dr} \tag{14}$$

to get

$$\Delta\phi = \frac{L/m}{r^2}\Delta\tau. \tag{15}$$

In order to derive the radial displacement Δr, the Schwarzschild proper time(i.e. from metric) is

$$\Delta s^2 = \Delta\tau^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \left(1 - \frac{2M}{r}\right)^{-1}\Delta r^2 - r^2\Delta\phi^2. \tag{16}$$

Since light is massless, one should take the limit $m \longrightarrow 0$. And, derive that

$$\Delta r = \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \left(1 - \frac{2M}{r}\right)\frac{(L/E)^2}{r^2}}\Delta t. \tag{17}$$

And,

$$r\Delta\phi = \pm\frac{L/E}{r}\left(1 - \frac{2M}{r}\right)\Delta t. \tag{18}$$

Insert (13) and (15) into (16), one gets

$$\Delta\tau^2 = \left(1 - \frac{2M}{r}\right)\left(\frac{E/m}{1 - \frac{2m}{r}}\right)^2\Delta\tau^2 - \frac{\Delta r^2}{(1 - \frac{2M}{r})} - r^2\left(\frac{L/m}{r^2}\right)^2\Delta\tau^2. \tag{19}$$

Simplify equation (19),

$$\Delta r = \pm\sqrt{\left(\frac{E}{m}\right)^2 - \left[1 + \left(\frac{L/m}{r}\right)^2\right]\left(1 - \frac{2M}{r}\right)\Delta\tau^2}. \tag{20}$$

Equation (20) is not the whole story unless the Taylor expansion w.r.t. τ be considered to second and more higher order for the case of falling into black holes.

For equation (20), the equation of motion can simply be written into the following form,

A more easier way to do is consider the effective potential.

$$A = B\dot{\vec{x}}^2 + V(x) \tag{21}$$

where A (equal to E/m in our example) and B (equal to 1/2 in our example) are constants (B being positive), $\vec{x}$ is the position vector of the object and $V(x)$ is the position dependent potential. With a *caveat*: equation (20) can be rewritten as

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right]. \tag{22}$$

Consider angular momentum of the photon

$$L = |\vec{r} \times \vec{p}| = rp\,sin\theta = pb \tag{23}$$

where $b = L/p = L/E$ which is the impact parameter.

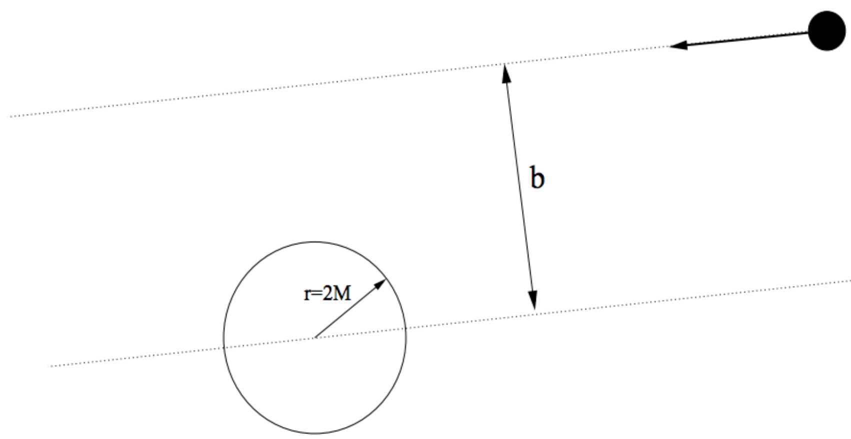

Figure 4: Definition of the impact parameter.

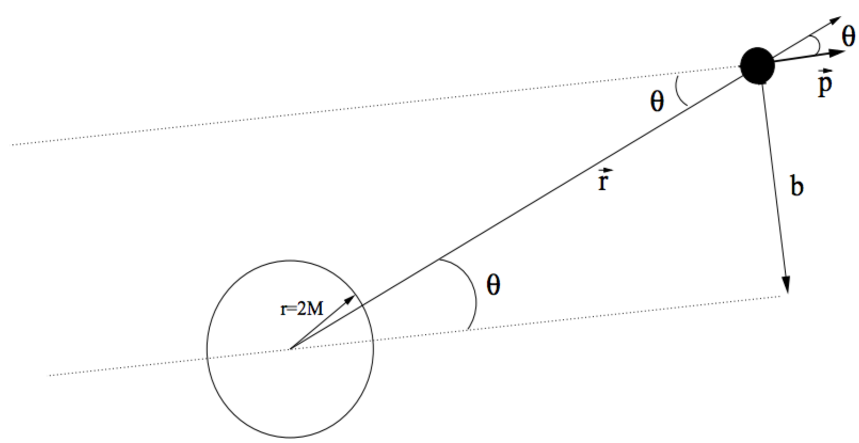

Figure 5: b be expressed in terms of L.

Therefore, b can be written as the ratio between angular momentum, L, and linear momentum, p.

$$b = \frac{L}{p}. \tag{24}$$

For a photon, $p = E$,

$$b = \frac{L}{E}. \tag{25}$$

According to equation (25), one can rewrite equation (17) and (18) into

$$\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \left(1 - \frac{2M}{r}\right)\left(\frac{b}{r}\right)^2} \tag{26}$$

And,

$$r\frac{d\phi}{dt} = \pm\frac{b}{r}\left(1 - \frac{2M}{r}\right). \tag{27}$$

One can use the equations of motion for a photon (see equation (26)) to show that the radial light speed dr_{shell}/dt_{shell} observed by a shell observer can take the following form:

$$\frac{1}{b^2}\left(\frac{dr_{shell}}{dt_{shell}}\right)^2 = \frac{1}{b^2} - \frac{\left(1 - \frac{2M}{r}\right)}{r^2}. \tag{28}$$

Deflection angle of light by a star is defined as following figure 6:

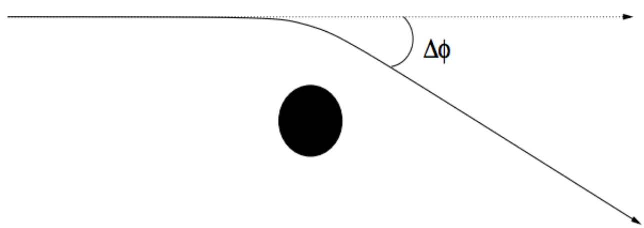

Figure 6: The dotted line is the direction light would have taken if no star, i.e., no deflection arise.

In more detail, to see the figure 7 below

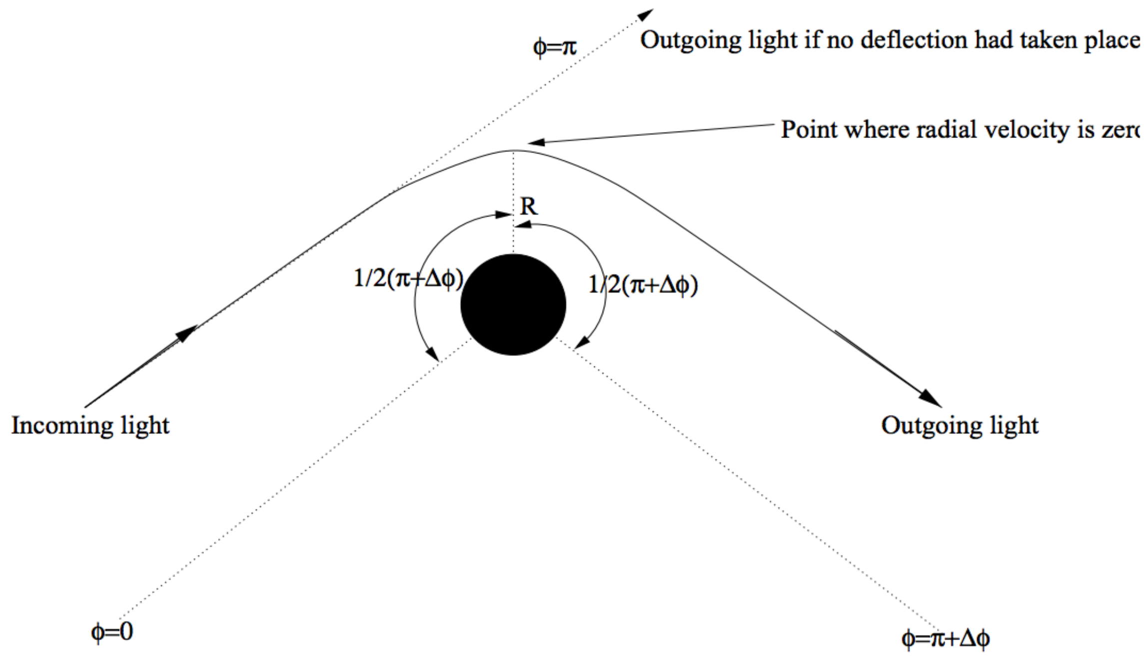

Figure 7: There is a symmetry that can let the situation equal on both side of the point where the distance between the light beam and the star is minimal $r = R$ and the radial velocity of the beam is zero.

According to equation of motion of the light in Schwarzschild geometry, i.e. (17) and (18), one can calculate the deflection angle, $\Delta\phi$, by diving each other of these two equations, then have the following

$$d\phi = \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}}}. \tag{29}$$

Integrating (29)

$$\int_0^{\frac{\pi}{2}+\frac{\Delta\phi}{2}} d\phi = \int_\infty^R \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r^2}}}. \tag{30}$$

Let $u = R/r$,

$$\int_0^{\frac{\pi}{2}+\frac{\Delta\phi}{2}} d\phi = \frac{1}{R}\int_0^1 \frac{du}{\sqrt{\frac{1}{b^2} - \left(\frac{u}{R}\right)^2\left(1 - \frac{2Mu}{R}\right)}}. \tag{31}$$

Consider equation (28), if radial velocity approaches to zero, i.e. on the left hand side, then one gets the following relation

$$\frac{1}{b^2} = \frac{\left(1 - \frac{2M}{r}\right)}{r^2}. \tag{32}$$

Then, substitute equation (32) into equation (31)

$$\frac{\pi}{2} + \frac{\Delta\phi}{2} = \int_0^1 \frac{du}{\sqrt{\left(1 - \frac{2M}{R}\right) - u^2\left(1 - \frac{2Mu}{R}\right)}}. \tag{33}$$

For stars, $R \gg 2M$, let $x := M/R, x \ll 1$. Then the integrand can be rewritten into the following form

$$f(x) = \left(1 - 2x - u^2(1 - 2xu)\right)^{-1/2}, \tag{34}$$

and considered by Taylor expansion,

$$f(x) \simeq f(0) + f'(0)x \tag{35}$$

where

$$f(0) = \frac{1}{\sqrt{1 - u^2}} \tag{36}$$

and

$$f'(0) = \frac{1 - u^3}{(1 - u^2)^{3/2}}. \tag{37}$$

Hence

$$\frac{\pi}{2} + \frac{\Delta\phi}{2} = \int_0^1 \frac{du}{\sqrt{1 - u^2}} + \frac{M}{R}\int_0^1 \left[\frac{1}{(1 - u^2)^{3/2}} - \frac{u^3}{(1 - u^2)^{3/2}}\right] du \tag{38}$$

$$= \frac{\pi}{2} + \frac{2M}{R}. \tag{39}$$

Therefore,

$$\frac{\Delta\phi}{2} = \frac{2M}{R}. \tag{40}$$

and we have already derived an important results:

$$\boxed{\Delta\phi = \theta_E = \frac{4M}{R} = \frac{4GM}{Rc^2} = 2\theta_N.} \tag{41}$$

where θ_E denotes Einstein's deflection angle, θ_N denotes Newton's deflection angle. This equation will be reconstructed in equation (53). However, only with the above derivation, one can really know the origin of the factor "2" in equation (53).

2.2 Derivation of Einstein Radius

2.2.1 Method I: A Quick Solution

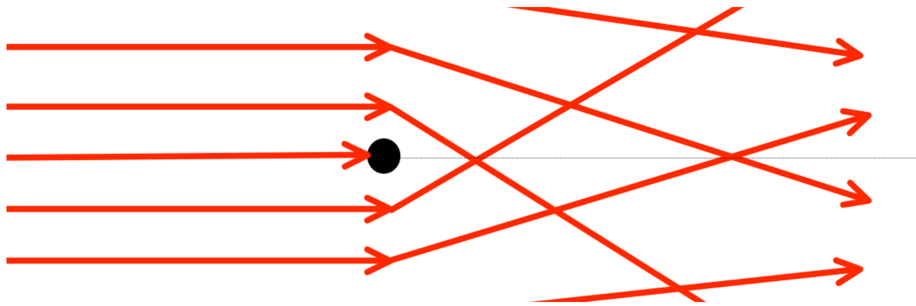

Figure 8: Analogue with Spherical Lens

According to equation (10) and equation (41), one can easily write down

$$\text{the Einstein bend angle } \theta_E = \frac{4GM}{bc^2}, \quad (42)$$

$$\text{and the Focal Length } f = \frac{b}{\theta_E} = \frac{b^2c^2}{4GM}. \quad (43)$$

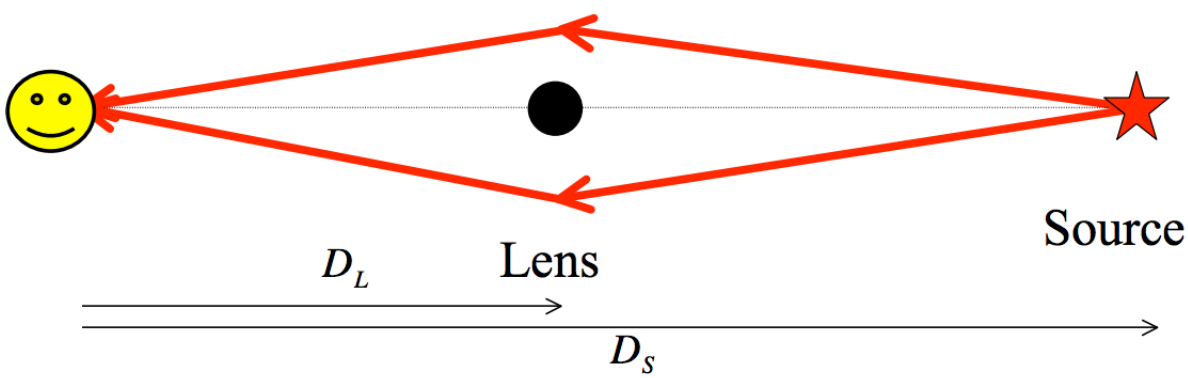

Figure 9: Geometric Optics Derivation of Einstein Ring Radius

By using geometric optics:

$$\frac{1}{D_S - D_L} + \frac{1}{D_L} = \frac{1}{f} = \frac{4GM}{c^2b^2} \quad (44)$$

According to equation (44), after simply rearrange the left had side one can easily derive the following results:

$$\begin{aligned}
\text{the Einstein radius } b &= R_E = \sqrt{\frac{4GM}{c^2}\frac{D_L D_{SL}}{D_S}}, &(45)\\
\text{the bend angle } \theta_E &= \frac{R_E}{D_L}, &(46)\\
&= \sqrt{\frac{4GM}{c^2}\frac{D_{SL}}{D_L D_S}}, &(47)\\
&= \left(\frac{M}{10^{11.1} M_{Sun}}\right)^{1/2} \left(\frac{D_L D_S / D_{LS}}{Gpc}\right)^{-1/2} arcsec. &(48)
\end{aligned}$$

2.2.2 Method II: A Rigorous Way

The effect of spacetime curvature on the light paths can then be expressed in terms of an effective index of refraction n, which is given by (e.g. Schneider et al. 1992)

$$n = 1 - \frac{2\Phi}{c^2} = 1 + \frac{2|\Phi|}{c^2}, \qquad (49)$$

As in the case of the prism(Shapiro 1964), light rays are deflected when they pass through a gravitational field. The deflection is the integral along the light path of the gradient of n perpendicular to the light path, i.e.

$$\hat{\alpha} = -\int \overrightarrow{\nabla_\perp} n dl = \frac{2}{c^2} \int \overrightarrow{\nabla_\perp} \Phi dl, \qquad (50)$$

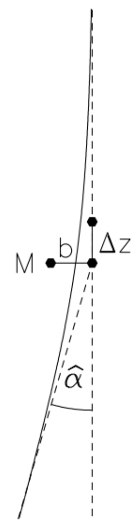

Figure 10: Light deflection by a point mass. Mind that most of the deflection occurs in the range that $\Delta z \approx \pm b$.

As an example, we now evaluate the deflection angle of a point mass M (cf. figure 10). The Newtonian potential of the lens is

$$\Phi(b, z) = \frac{-GM}{(b^2 + z^2)^{1/2}} \tag{51}$$

where b is the impact parameter of the unperturbed light ray, and z indicates distance along the unperturbed light ray from the point of closest approach. We therefore have

$$\overrightarrow{\nabla_\perp}\Phi(b, z) = \frac{GM}{(b^2 + z^2)^{3/2}} \tag{52}$$

where $\vec{b}$ is orthogonal to the unperturbed ray and points toward the point mass. Equation (52) then yields the deflection angle

$$\hat{\alpha} = \frac{2}{c^2}\int \overrightarrow{\nabla_\perp}\Phi dz = \frac{4GM}{c^2 b}. \tag{53}$$

Thus, we reconstruct equation (41) and equation (43) again. The mass distribution of the lens can then be projected along the line-of-sight and be replaced by a *mass sheet* orthogonal to the line-of-sight. The plane of the mass sheet is commonly called the *lens plane*. The mass sheet is characterized by its surface mass density

$$\Sigma(\vec{\xi}\,) = \int \rho(\vec{\xi}, z) dz \tag{54}$$

where $\vec{\xi}$ is a two-dimensional vector in the lens plane. The deflection angle at position, ξ, is the sum of the deflections due to all the mass elements in the plane:

$$\vec{\hat{\alpha}} = \frac{4G}{c^2} \int \frac{\Sigma(\vec{\xi'})(\vec{\xi} - \vec{\xi'})}{\|\vec{\xi'} - \vec{\xi}\|^2} d^2\xi'. \tag{55}$$

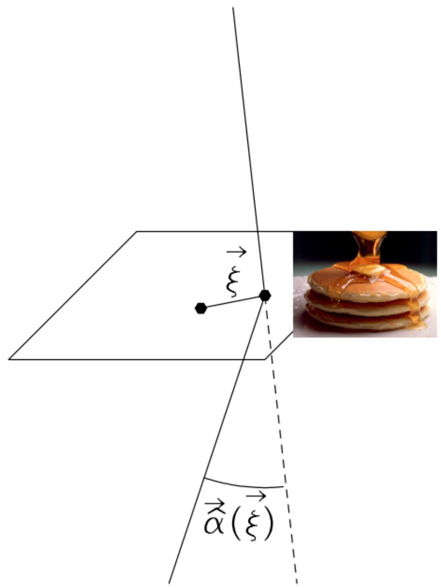

Figure 11: If the distance between observer and lens which is larger larger than Δz than the spherical star can be squeeze like a "pancake", i.e. this *pancake (thin lens)* usually be called the lens plane. As we project all mass into lens plane, the notation of the impact parameter, $\vec{b}$, also change into $\vec{\xi}$.

In general, the deflection angle is a two-component vector. For some special cases with circularly symmetric lens, one can shift the coordinate origin to the center of symmetry and reduce light deflection to a one-dimensional problem. The deflection angle then points toward the center of symmetry, and its modulus is taken the following form:

$$\hat{\alpha}(\xi) = \frac{4GM(\xi)}{c^2\xi} \tag{56}$$

where ξ is the distance from the lens center and $M(\xi)$ is the mass enclosed within radius ξ,

$$M(\xi) = 2\pi \int_0^\xi \Sigma(\xi')\xi' d\xi'. \tag{57}$$

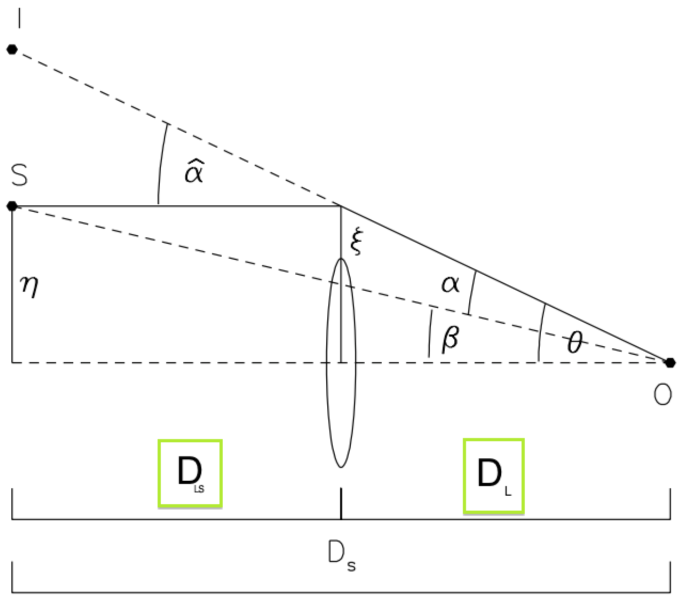

Figure 12: An Illustration of A Gravitational Lens System.

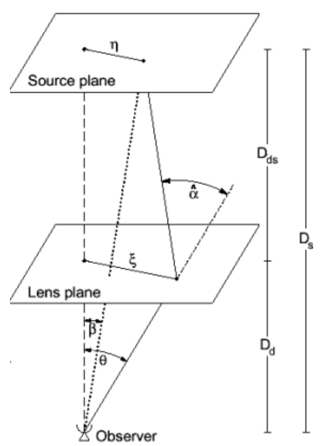

Figure 13: A 3D Illustration

For the convenient, we introduce the reduced deflection angle:

$$\vec{\alpha} = \frac{D_{LS}}{D_S}\vec{\hat{\alpha}}. \tag{58}$$

From figure (12) and figure (13) one can see that

$$\because \theta D_S = \beta D_S + \hat{\alpha} D_{LS}. \tag{59}$$

Thus, the positions of the source and the image are related through the simple equation

$$\therefore \vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}). \tag{60}$$

Distances so defined are called angular-diameter distances, and equation (58) and (59) are valid only when these distances are used. Equation (60) is called the lens equation, or ray-tracing equation. It is nonlinear in the general case, and so it is possible to have multiple images described by $\vec{\theta}$ corresponding to a single source position, denotes $\vec{\beta}$. As Figure 12 shows, the lens equation is trivial to derive and requires merely that the following Euclidean relation should exist between the angle enclosed by two lines and their separation,

Note that in general $D_{LS} \neq D_S - D_L$.

$$Separation = angle \times distance. \tag{61}$$

As an instructive special case consider a lens with a constant surface-mass density. From equation (56), the reduced deflection angle is

$$\alpha(\theta) = \frac{D_{LS}}{D_S}\frac{4G}{c^2\xi}\left(\Sigma\pi\xi^2\right) = \frac{4\pi G\Sigma}{c^2}\frac{D_L D_{LS}}{D_S}\theta \tag{62}$$

where we have set $\xi = D_L\theta$. In our case, the lens equation is linear; that is, $\beta \propto \theta$. Let's define a critical surface-mass density

$$\Sigma_{cr} = \frac{c^2}{4\pi G}\frac{D_S}{D_L D_{LS}} = 0.35\text{g cm}^{-2}\left(\frac{D}{1Gpc}\right)^{-1}. \tag{63}$$

where the effective distance D is defined as the combination of distances

$$D = \frac{D_L D_{LS}}{D_S}. \tag{64}$$

For a lens with a constant surface mass density Σ_{cr}, the deflection angle is $\alpha(\theta) = \theta$. Thus, $\beta = 0$ for all θ. In optics, a lens in this kind usually focuses perfectly, with a well-dened focal length(However, in gravitational lensing is not the case. Instead, light rays which pass the lens at different impact parameters cross the optic axis at different distances behind the lens and also with different wavelengths.

A lens which has $\Sigma > \Sigma_{cr}$ somewhere within it is referred to as being *supercritical*(here is the definition of *supercritical*). Usually, multiple imaging occurs only if the lens is supercritical, but there are exceptions to this rule (e.g., Subramanian & Cowling 1986). Consider now a circularly symmetric lens with an arbitrary mass. According to equation 56, 58 and 60, the lens equation have the following form:

$$\beta(\theta) = \theta - \frac{D_{LS}}{D_L D_S}\frac{4GM(\theta)}{c^2\theta}. \tag{65}$$

Owning to the rotational symmetry of the lens system, a source which lies exactly on the optic axis, with $\beta = 0$, is imaged as a ring if the lens is *supercritical*. Therefore, after setting $\beta = 0$ in equation (65), and with a simple rearrangement on both sides, we have already derived the equation (47) again.

$$\boxed{\theta_E = \left[\frac{4GM(\theta_E)}{c^2}\frac{D_{LS}}{D_L D_S}\right]^{1/2}} \tag{66}$$

A Dictionary For All Critical Physical Quantities And Constants	
c	velocity of light
G	Newton constant
L	angular momentum of the star-photon system
n	effective index of refraction
Φ	the Newtonian potential of the lens
z	the distance along the unperturbed light ray from the point of closest approach
D_S	distance of observer-source
D_L	distance of observer-lense
D_{LS}	distance of lense-source
D	effective distance, see equation (64)
R_E, b	Einstein radius and impact parameter
$\Delta\phi$	deflection angle between incoming and outgoing light(photon)
θ_E	Einstein radius in bend angle expression; Einsteins deflection angle
θ_E	Newtons deflection angle
$\vec{\xi}$	a two-dimensional vector in the lens plane, its modulus, ξ, is the distance from the lens center, impact parameter in $\Delta z \ll D_S$ limit
θ	angular distance between image-center
$\alpha(\theta)$	apparent angular deflection
$\hat{\alpha}$	equal to θ_E, note that the Schwarzschild radius of a point mass is $2GM/c^2$, so that the deflection angle is simply twice the inverse of the impact parameter in units of the Schwarzschild radius. sometimes also called "actual deflection of the light ray."
$\beta(\theta)$	angular distance between source-center
Σ	surface-mass density
Σ_{cr}	critical surface-mass density, see equation (63)
M_{Sun}	mass of the Sun
$M(\xi), M(\theta)$	the mass enclosed within radius ξ
η	a distance on source plane which is between source-center