A Study on Extended Theories of Gravity(ETGs)

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Outline

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- A Quick Overview and Introduction
- Setup of Models
- Outlook and Future Work

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Comparison Table

$\kappa^2 = 8\pi G$ Intro. of Conformal Transformation

Gravity Models	Action $\mathcal{S}$ and Field Equations
ΛCDM or GR with CC	$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R - 2\Lambda) + \int d^4x\,\mathcal{L}_M$ $G_{\mu\nu} = \kappa^2 T_{\mu\nu} - \Lambda g_{\mu\nu}$
Scalar-Tensor Theories	$\mathcal{S} = \int d^4x\sqrt{-g}[f(\phi)R - \frac{1}{2}h(\phi)g^{\mu\nu}\nabla_\mu(\phi)\nabla_\nu\phi - U(\phi)] + \int d^4x\sqrt{-g}\mathcal{L}_M(g^{\mu\nu}, \psi_i)$ $G_{\mu\nu} = f^{-1}(\phi)(\frac{1}{2}T^{(M)}_{\mu\nu} + g_{\mu\nu}f\frac{1}{2}T^{(\phi)}_{\mu\nu} + \nabla_\mu\nabla_\nu\Box f)$

Ref. Dunsby P. K., et. al. "ΛCDM universe in f(R) gravity", *Phys. Rev.* D**82** 023519 (2010)

Comparison Table

Gravity Models	Action $\mathcal{S}$ and Field Equations
f(R)-Metric formalism	$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x \sqrt{-g} f(R) + \int d^4x\, \mathcal{L}_M(g_{\mu\nu}, \Psi_M)$ $G_{\mu\nu} = \kappa^2 (T^{(M)}_{\mu\nu} + T^{(D)}_{\mu\nu})$
f(R)-Palatini formalism	$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(\tilde{R}) + \int d^4x\, \mathcal{L}_M(g_{\mu\nu}, \Psi_M)$ $f'(\tilde{R})\tilde{R}_{\mu\nu} - \frac{f(\tilde{R})}{2} g_{\mu\nu} = \kappa T_{\mu\nu}$

Comparison Table

Gravity Models	Action $\mathcal{S}$
Galileon gravity	$\mathcal{S} = \int d^D x \sqrt{-g} \sum_{p=0}^{p_{max}} \mathcal{C}_{(n+1,p)} \mathcal{L}_{(n+1,p)}$,

$$\mathcal{L}_{(n+1,p)} = -\frac{1}{(D-n)!} \varepsilon^{\mu_1 \mu_3 \cdots \mu_{2n-1} \nu_1 \cdots \nu_{(D-n)}} \varepsilon^{\mu_2 \mu_3 \cdots \mu_{2n} \nu_1 \cdots \nu_{(D-n)}}$$

$$\times \pi_{;\mu_1} \pi_{;\mu_2} (\pi^{;\lambda} \pi_{;\lambda})^p \prod_{i=1}^{p} R_{\mu_{4i-1} \mu_{4i+1} \mu_{4i} \mu_{4i+2}} \prod_{j=0}^{n-2-2p} \pi_{;\mu_{2n-1-2j} \mu_{2n-2j}}.$$

$$\mathcal{C}_{(n+1,p)} = (\frac{-1}{8})^p \frac{(n-1)!}{(n-1-2p)!(p!)^2}$$

p_{max} is the integer part of $\frac{(n-1)}{2}$, $n \leq D$.

Ref. [1]Deffayet, C., Deser, S., and Esposito-Farèse, G., "Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stresstensors", *Phys. Rev.* D,**80**, 064015, (2009).
Ref. [2] Nicolis, A., Rattazzi, R., and Trincherini, E., "Galileon as a local modification of gravity" *Phys. Rev.* D,**79**, 064036, (2009).

Comparison Table

Gravity Models	Action $\mathcal{S}$ and Field Equations
Gauss-Bonnet	$\mathcal{S} = \frac{1}{\kappa^2}\int d^4x\,\sqrt{-g}(\frac{R}{2} + f(\mathcal{G})) + \int d^4x\,\mathcal{L}_M$ $G_{\mu\nu} + 8[R_{\mu\rho\mu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho}$ $+\frac{R}{2}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})]\nabla^{\rho}\nabla^{\sigma}f_{,\mathcal{G}} + (\mathcal{G}f_{,\mathcal{G}} - f)g_{\mu\nu} = \kappa^2 T_{\mu\nu}$ $\mathcal{G} := R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$

Comparison Table

Go to Def. of Cotton Tensor

Gravity Models	Action $\mathcal{S}$ and Field Equations
3+1,non-dynamical CS theory	$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\, \sqrt{-g}R + \frac{1}{8\kappa^2}\int d^4x\, \sqrt{-g}\vartheta\, {}^*RR$ $G_{\mu\nu} + C_{\mu\nu} = \kappa^2 T_{\mu\nu}$ ${}^*RR = 0$
3+1,dynamical CS theory	$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\, \sqrt{-g}R - \frac{\ell}{12}\int d^4x\, \sqrt{-g}\,(\vartheta {}^*RR)$ $+\frac{\ell}{2}\int d^4x\, \sqrt{-g}[\, g^{\mu\nu}(\nabla_\mu\vartheta)(\nabla_\nu\vartheta) + 2V(\vartheta)\,]$ $G_{\mu\nu} + \frac{-\ell}{3}16\pi G C_{\mu\nu} = \kappa^2 T_{\mu\nu}$ $\beta\Box\vartheta = \beta\frac{dV}{d\vartheta} - \frac{-\ell}{12}({}^*RR)$

Comparison Table

Gravity Models	Action $\mathcal{S}$ and Field Equations
Hořava-Lifshitz(HL) Gravity **detailed-balanced**	$S = \int dt\, d^3x\, \sqrt{g} N \{ \frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2}{2W} C_{ji} C^{ij} - \frac{\kappa^2 \mu \varepsilon^{ijk}}{2W^2 \sqrt{g}} R_{i\ell} \nabla_j R^l_k + \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu^2}{8(1-3\lambda)} [\frac{1-4\lambda R^2}{4}] + \Lambda R - 3\Lambda^2 \}$

Introduction

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Question:

Since GR is accepted to describe the geometric properties of spacetime and the standard big-bang cosmology model can be well described within the framework of GR, why we still need to extend or modify it?
i.e. Why we need ETG/MG?

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i.e. Why we need ETG/MG?

Reason:

Observational Cosmology shows taht the universe has undergone two phases of cosmic acceleration.

Introduction
A Capsul History

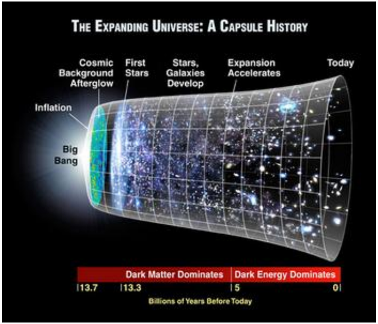

Figure: The Expanding Universe-A Capsul History

Introduction

A Capsul History

The First Phase: Inflation

Inflation have occurred prior to the radiation domination.
This phase is required to solve following problems:
1) flatness and horizon problems
2) nearly flat spectrum of T anisotropies in CMB

1. Guth, Z.-K., Otha, N., and Tsujikawa, S., "Realizing scale-invariant density perturbations in low-energy effective string theory" , *Phys. Rev. D***75**, 023520, (2007).
2. Bassett, B.A., and Liberati, S., "Geometric reheating after inflation", *Phys. Rev. D*,**58**, 021302, (1998).

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A Capsul History

The Second Accelerating Phase

Started after the matter domination.
This phase have unknown component which giving rise to this late-time cosmic acceleration is called dark energy.
Then, what is DE?

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- Huterer, D., and Turner, M.S., ”Prospects for probing the dark energy via supernova distance measurements” , *Phys. Rev. D***60**, 081301, (1999). [astro-ph/9808133]

Introduction

Observations About DE

Dark Energy

DE has been probed by the following observations:
1) Type Ia Supernova(SN Ia)
2) large-scale structure(LSS)
3) CMB Anisotropy
4) Baryon Acoustic Oscillations(BAO)

LSS Tegmark, M. et al. (SDSS Collaboration), ”Cosmological parameters from SDSS and WMAP” , *Phys. Rev. D***69**, 103501, (2004).

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What we understand about Dark Energy?

1. Smooth, very elastic, non-particulate medium can be analog with perfect fluid

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Ref. Pisin Chen's Lecture Note, "Lecture 19 Dark Energy II: The Theoretical Challenge" (2010)

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DE touches many other important puzzles

1. Vacuum Energy, Cosmological Constant

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Several Approach to Explain DE$\Longrightarrow$ETG/MG is One Them

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2. Quintessence (Scalar Field) as DE $\Rightarrow$Assume GR is correct
   Model 1: Exponential potentials of quintessence $\Rightarrow$ is ruled out at the 95.4% confidence level (arXiv:0903.2423)
   Model 2: Power-law potentials
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   Model 5: ....etc.

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Je-An Gu, Chien-Wen Chen, Pisin Chen, New J. Phys. 11: 073029, (2009)

Introduction

Several Approach to Explain DE$\Longrightarrow$ETG/MG is One Them

In the end, ETG can explain DE in phenomenological way, furthermore, ETG can also explain the dark matter(DM) problem in phenomenological way. But, here, let's foucs on DE Prob. and analyze by using ETGs'.

Setup of the Model
(0) Mathematical Tool

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(1) Scalar-Tensor Theories
(2) F(R) Theories
(3) Galileon Gravity
(4) Gauss-Bonnet Gravity
(5) Chern-Simons(CS) Theories
(6) Hořava-Lifshitz(HL) Gravity
(7) Entropic Forces

(5) Mathematical Tool

(5) Mathematical Tool

1. Cotton Tensor

(5) Mathematical Tool

1. Cotton Tensor
2. Conformal Transformation

Mathematical Tool— Cotton Tensor

Both in CS theories and Hořava-Lifshitz(HL) Gravity, we all need to use Cotton tensor. The following is a brief review: In differential geometry, the Cotton tensor on a (pseudo)-Riemannian manifold of dimension n is a third-order tensor conomitant of the metric, like the Weyl tensor. Just as the vanishing of the Weyl tensor for $n \geq 4$ is a necessary and sufficient condition for the manifold to be conformally flat, the same is true for the Cotton tensor for $n = 3$, while for $n \leq 2$ it is identically zero.

Mathematical Tool— Cotton Tensor

The proof of the classical result that for n=3 the vanishing of the Cotton tensor is equivalent the metric being conformally flat is given by Eisenhart using a standard integrability argument. This tensor density is uniquely characterized by its conformal properties coupled with the demand that it be differentiable for arbitrary metrics, as shown by Aldersley.

Mathematical Tool— Cotton Tensor

Cotton Tensor

In coordinates, and denoting the Ricci tensor by R_{ij} and the scalar curvature by R, the components of the Cotton tensor are $C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}(\nabla_j R g_{ik} - \nabla_k R gij)$ sometimes called the Cotton –York tensor.

Mathematical Tool— Cotton Tensor

Cotton-York Tensor

The Cotton tensor can be regarded as a vector valued 2-form, and for n=3 one can use the Hodge star operator to convert this in to a second order trace free tensor density

$C^{\imath}_{\jmath} = \nabla_k (R_{\ell\imath} - \frac{1}{4} R g_{\ell\imath}) \epsilon^{k\ell\jmath}$ or $C^{ij} = \frac{\epsilon^{ijK}}{\sqrt{g}} (R^j_i - \frac{1}{4} R \delta^i_j)$

Back to Comparison Table

Conformal Transformation —Applications

- 1984 Wald
  GR: Asymptotic flatness and in the initial value formulation.
- Fermat Principal
  Perlick 1990, Schneider, Ehlers and Falco 1992
- Gravitational Lensing in the (conformally flat)
  Friedmann-Lemaitre-Robertson-Walker Universe
  Perlick 1990, Schneider, Ehlers and Falco 1992

Conformal Transformation —Applications

- Wave Equations
  Sonego and Faraoni 1992
  Nonan 1995
- Optical Geometry near black hole horizons
  Abramowicz, Carter and Lasota 1988
  Sonego and Massar 1997
  Abramowicz et al. 1997
- Exact Solutions
  Van der Bergh 1986, 1988

Conformal Transformation —Applications

- Quantum Field Theory in Curved Spaces
  Birell and Davies 1982
- Statistical Mechanics and String Theories
  Dita and Georgescu 1989

Conformal Transformation —Intro.

which is often used as a mathematical tool to map the equations of motion of physical systems into mathematically equivalent sets of equations that are more easily solved and computationnally more convenient to study.

Conformal Transformation —Intro.

which is mainly used in three different areas of gravitational physics:

1. Alternative (including nonlinear) theories of gravity

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1. Alternative (including nonlinear) theories of gravity
2. Unified theories in multi-dimensional spaces
3. Scalar fields non-minimally coupled to gravity

Conformal Transformation —Intro.

more precisely description

A conformal transformation is essentially a local change of scale.

Conformal Transformation —Intro.

more precisely description

Since the distance are measured by the metric, hence the metric is multiplied by a spacetime-dependent (nonvanishing) function.

$\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}$

i.e. $\tilde{d}^2 s = \omega^2(x) d^2 s$

for some nonvanishing function $\omega(x)$ x is used to denote the collection of spacetime coordinate x^μ.

Note that

$g_{\mu\nu} = \omega^{-2}(x)\tilde{g}_{\mu\nu}$ is trivial.

Conformal Transformation —Intro.

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with this sense we have two way to appy

Applications

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Applications

1. To change dynamical variables in a scalar-tensor theories.

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Applications

1. To change dynamical variables in a scalar-tensor theories.
2. Remap spacetimes into convenient conformal (Penrose)diagram.

Conformal Transformation —A Critical Fact

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Null curves are left invariant by conformal transformations.

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Theorem

A curve $x^\mu(x)$ is null iff. its tangent vector $\frac{dx^\mu}{d\lambda}$ is null.

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Extension—Property

By the theorem above we can show that "Conformal transformation leave light cones invariant."

Conformal Transformation —A Critical Fact

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Proof of Extended Property

1. By the Thm.

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2. $g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} = 0$

Conformal Transformation —A Critical Fact

Proof of Extended Property

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2. $g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0$
3. Then by the conformal transformation, we have the relation such as:
   $\tilde{g}_{\mu\nu} = \omega^2(x)g_{\mu\nu}$
   substisude into above equation

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e.g. Timelike geodesics of $\tilde{g}_{\mu\nu}$ will generally differ from timelike geodesics of $g_{\mu\nu}$.

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Anything that is function of $g_{\mu\nu}$ can be equally well thought of as a function of $\tilde{g}_{\mu\nu}$ and $\omega(x)$.

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Anything that is function of $g_{\mu\nu}$ can be equally well thought of as a function of $\tilde{g}_{\mu\nu}$ and $\omega(x)$.
Then it can be called that the quantities are expressed in the conformal frame.

Conformal Transformation

Christoffel Symbols

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \frac{1}{\omega}(\delta^{\rho}_{\mu}\nabla_{\nu}\omega + \delta^{\rho}_{\nu}\nabla_{\mu}\omega - g_{\mu\nu}g^{\rho\lambda}\nabla_{\lambda}\omega)$$

Conformal Transformation

Christoffel Symbols

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$$= \Gamma^{\rho}_{\mu\nu} + C^{\rho}_{\mu\nu}$$

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where $C^{\rho}_{\mu\nu} := \frac{1}{\omega}(\delta^{\rho}_{\mu}\nabla_{\nu}\omega + \delta^{\rho}_{\nu}\nabla_{\mu}\omega - g_{\mu\nu}g^{\rho\lambda}\nabla_{\lambda}\omega)$

Conformal Transformation

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The first covariant derivative of a scalar field ϕ

$\widetilde{\nabla}_{\mu}\phi$

Conformal Transformation

The first covariant derivative of a scalar field ϕ

$$\widetilde{\nabla}_\mu \phi = \nabla_\mu \phi$$

Conformal Transformation

The first covariant derivative of a scalar field ϕ

$$\widetilde{\nabla}_{\mu}\phi = \nabla_{\mu}\phi = \partial_{\mu}\phi$$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

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The second covariant derivative of a scalar field ϕ

In Conformal Frame

$$\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \phi$$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

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Conformal Transformation

The second covariant derivative of a scalar field ϕ

In Conformal Frame

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\nabla_\alpha \omega)(\nabla_\beta \phi)$$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

In Conformal Frame

$$\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\nabla_\alpha \omega)(\nabla_\beta \phi)$$

In Original Frame

$$\nabla_\mu \nabla_\nu \phi$$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

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In Original Frame

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In Conformal Frame

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\nabla_\alpha \omega)(\nabla_\beta \phi)$$

In Original Frame

$$\nabla_\mu \nabla_\nu \phi = \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\tilde{\nabla}_\alpha \omega)(\tilde{\nabla}_\beta \phi)$$

Conformal Transformation

D' Alembertian

In Conformal Frame

$\widetilde{\Box}\phi$

Conformal Transformation

D' Alembertian

In Conformal Frame

$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi$$

Conformal Transformation

D' Alembertian

In Conformal Frame

$$\tilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\omega)(\nabla_\beta\phi)$$

Conformal Transformation

D' Alembertian

In Conformal Frame

$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\omega)(\nabla_\beta\phi)$$

In Original Frame

$$\Box\phi$$

Conformal Transformation

D' Alembertian

In Conformal Frame

$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\omega)(\nabla_\beta\phi)$$

In Original Frame

$$\Box\phi = \omega^2\widetilde{\Box}\phi$$

Conformal Transformation

D' Alembertian

In Conformal Frame

$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\omega)(\nabla_\beta\phi)$$

In Original Frame

$$\Box\phi = \omega^{2}\widetilde{\Box}\phi - (n-2)\tilde{g}^{\alpha\beta}\omega(\widetilde{\nabla}_\alpha\omega)(\widetilde{\nabla}_\beta\omega)$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$\widetilde{R}$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega)$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

In Original Frame

$$R$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

In Original Frame

$$R = \omega^2 \widetilde{R}$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

In Original Frame

$$R = \omega^2 \widetilde{R} + 2(n-1) \tilde{g}^{\alpha\beta} \omega (\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \omega)$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2}R - 2(n-1)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\nabla_\beta\omega) - (n-1)(n-4)g^{\alpha\beta}\omega^{-4}(\nabla_\alpha\omega)(\nabla_\beta\omega)$$

In Original Frame

$$R = \omega^2\widetilde{R} + 2(n-1)\tilde{g}^{\alpha\beta}\omega(\widetilde{\nabla}_\alpha\widetilde{\nabla}_\beta\omega) - n(n-1)\tilde{g}^{\alpha\beta}(\widetilde{\nabla}_\alpha\omega)(\widetilde{\nabla}_\beta\omega)$$

(0) Mathematical Tool: Conformal Transformation

Reference[1]

Sean Carroll, ”SPACE AND GEOMETRY: An Introduction to General Relativity” Addison Wesley,(2004)

Reference[2]

Weinberg, S ”Gravitation and Cosmology” John Wiley, New York N.Y. (1972)

(0) Mathematical Tool: Conformal Transformation

Reference[3]

Dabrowski M. P., Garecki J, Blaschke D. B, Ann. Phys. **18**, 13

Reference[4]

Sotiriou T. P., Faraoni V., ”f(R) theories of gravity” Rev. Mod. Phys. **82**, 451(2010)

(0) Mathematical Tool: Conformal Transformation

Reference[5]

Faraoni V., Gunzig E. and Nardone P., ”Conformal transformations in classical gravitational theories and in cosmology”, Fundamentals of Cosmic Physics, arXiv: gr-qc 9811047 (1998)

Reference[6]

Felice A. D., Tsujikawa S., ”f(R) Theories” arXiv: gr-qc 1002.4928 (2010)

Back to the Comparison Table

Setup of the Model

(1) Scalar-Tensor Theories

(0) Mathematical Tool
(1) Scalar-Tensor Theories
(2) f(R) Theories
(3) Galileon Gravity
(4) Gauss-Bonnet Gravity
(5) Chern-Simons(CS) Theories
(6) Hořava-Lifshitz(HL) Gravity
(7) Entropic Forces

In theoretical physics, a scalar-tensor theory is a theory that includes both a scalar field and a tensor field to represent a certain interaction. For example, the Brans-Dicke theory of gravitation uses both a scalar field and a tensor field to mediate the gravitational interaction.
The general form of action is as following:
$$\mathcal{S} = \int d^4x\sqrt{-g}[f(\phi)R - \frac{1}{2}h(\phi)g^{\mu\nu}\nabla_\mu(\phi)\nabla_\nu\phi - U(\phi)] + \int d^4x\sqrt{-g}\mathcal{L}_M(g^{\mu\nu},\psi_i)$$

In general, we have the field equation in this following form
$$G_{\mu\nu} = f^{-1}(\phi)(\frac{1}{2}T^{(M)}_{\mu\nu} + g_{\mu\nu}f\frac{1}{2}T^{(\phi)}_{\mu\nu} + \nabla_\mu\nabla_\nu\Box f)$$

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Comparison Table

Setup of the Model

(2) f(R) Theories

(0) Mathematical Tool
(1) Scalar-Tensor Theories
(2) f(R) Theories
(3) Galileon Gravity
(4) Gauss-Bonnet Gravity
(5) Chern-Simons(CS) Theories
(6) Hořava-Lifshitz(HL) Gravity
(7) Entropic Forces

f(R) Theories

In f(R) theories, so far we have three formalism:

f(R) Theories

In f(R) theories, so far we have three formalism:

1. Metric formalism

f(R) Theories

In f(R) theories, so far we have three formalism:

1. Metric formalism
2. Palatini formalism

f(R) Theories

In f(R) theories, so far we have three formalism:

1. Metric formalism
2. Palatini formalism
3. Metric-Affine formalism

Metric formalism

$T^{(M)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}$

$T^{(D)}_{\mu\nu} := \frac{1}{\kappa^2}\left(\frac{g_{\mu\nu}(f-R)}{2} + \nabla_\mu \nabla_\nu F - g_{\mu\nu}\Box F + (1-F)R_{\mu\nu}\right)$

Since $\nabla^\mu G_{\mu\nu} = 0$ and $\nabla^\mu T^{(M)}_{\mu\nu} = 0$

It follows that $\nabla^\mu T^{(D)}_{\mu\nu} = 0$

Comparison Table

Palatini formalism

In this approach the action is varied with respect to both the metric and the connection. Unlike the metric approach and are treated as independent variables. Variations using the Palatini approach, lead to second-order field equations which are free from the instability associated with negative signs. We note that even in 1930's Lanczos proposed a specific combination of curvature-squared terms that lead to a secondorder and divergence-free modified Einstein equation.

Palatini formalism

Ref.

Palatini formalism

Ref.
[1]Ferraris M., Francaviglia M. and Volovich, I. ”The universality of vacuum Einstein equations with cosmological constant”, *Class. Quantum Grav.*, **11**, 1505, (1994)

Palatini formalism

Ref.

[1]Ferraris M., Francaviglia M. and Volovich, I. "The universality of vacuum Einstein equations with cosmological constant", *Class. Quantum Grav.*, **11**, 1505, (1994)

[2]Vollick, D.N., "1\R curvature corrections as the source of the cosmological acceleration", *Phys. Rev. D*,**68**, 063510, (2003)

Comparison Table

Setup of the Model

(3) Galileon Gravity

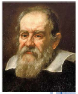
(0) Mathematical Tool
(1) Scalar-Tensor Theories
(2) f(R) Theories
(3) Galileon Gravity
(4) Gauss-Bonnet Gravity
(5) Chern-Simons(CS) Theories
(6) Hořava-Lifshitz(HL) Gravity
(7) Entropic Forces

Galileon Symmetry

$\pi \to \pi + c + b_\mu x^\mu$

where c and b_μ are constants, we call π a Galileon field

Galileon gravity



The field self-interaction $\Box\phi(\partial_\mu\phi\partial^\mu\phi)$ appearing in the DGP model satisfies the Galilean symmetry in the flat space-time:

$$\partial_\mu\phi \rightarrow \partial_\mu\phi + b_\mu$$

What kind of Lagrangians do we get by imposing the Galilean symmetry?

In the flat space-time Nicolis et al. (2008) showed that there are five Lagrangians in total including

$$\mathcal{L}_1 = \phi, \quad \mathcal{L}_2 = (\nabla\phi)^2, \quad \mathcal{L}_3 = (\Box\phi)(\nabla\phi)^2$$

In the curved space-time the Galilean symmetry is in general broken.

One can construct covariant Lagrangians that recover the Galilean symmetry in the limit of the flat space-time.

S. Tsujikawa"Cosmology of a Galileon Field" presentation slide

Covariant Galileons

Deffayet et al. (2009)

There are five covariant Lagrangians that respect the Galilean symmetry in the Minkowski spacetime:

$$\mathcal{L}_1 = M^3\phi\,, \quad \mathcal{L}_2 = (\nabla\phi)^2\,, \quad \mathcal{L}_3 = (\Box\phi)(\nabla\phi)^2/M^3\,,$$

$$\mathcal{L}_4 = (\nabla\phi)^2\left[2(\Box\phi)^2 - 2\phi_{;\mu\nu}\phi^{;\mu\nu} - R(\nabla\phi)^2/2\right]/M^6,$$

$$\mathcal{L}_5 = (\nabla\phi)^2[(\Box\phi)^3 - 3(\Box\phi)\,\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu}{}^{\nu}\phi_{;\nu}{}^{\rho}\phi_{;\rho}{}^{\mu} - 6\phi_{;\mu}\phi^{;\mu\nu}\phi^{;\rho}G_{\nu\rho}]/M^9$$

M: some mass scale

The above Lagrangians are constructed to keep the equations of motion up to the second-order.

- **How about the cosmology based on the above Lagrangians ?**
- **The ghost does not appear unlike the DGP model?**

S. Tsujikawa"Cosmology of a Galileon Field" presentation slide

Galileon cosmology

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm pl}^2}{2} R + \frac{1}{2} \sum_{i=1}^{5} c_i \mathcal{L}_i \right] + S_m$$

$\mathcal{L}_i \quad (i = 1, \cdots, 5)$: five covariant Galileon Lagrangians

We set $c_1 = 0$ to discuss the case in which the late-time cosmic acceleration can be realized by the field kinetic energy.

The equations of motion in the flat FLRW background (in the presence of non-relativistic matter and radiation):

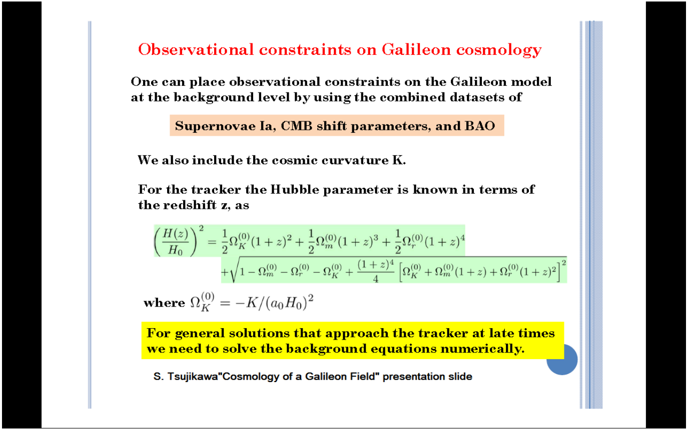
$$3M_{\rm pl}^2 H^2 = \rho_{\rm DE} + \rho_m + \rho_r$$
$$3M_{\rm pl}^2 H^2 + 2M_{\rm pl}^2 \dot{H} = -P_{\rm DE} - \rho_r/3$$

where

$$\rho_{\rm DE} \equiv -c_2\dot{\phi}^2/2 + 3c_3 H\dot{\phi}^3/M^3 - 45c_4 H^2\dot{\phi}^4/(2M^6) + 21c_5 H^3\dot{\phi}^5/M^9$$
$$P_{\rm DE} \equiv -c_2\dot{\phi}^2/2 - c_3\dot{\phi}^2\ddot{\phi}/M^3 + 3c_4\dot{\phi}^3[8H\ddot{\phi} + (3H^2 + 2\dot{H})\dot{\phi}]/(2M^6)$$
$$-3c_5 H\dot{\phi}^4[5H\ddot{\phi} + 2(H^2 + \dot{H})\dot{\phi}]/M^9 .$$

(second-order)

S. Tsujikawa"Cosmology of a Galileon Field" presentation slide


Observational constraints on Galileon cosmology
One can place observational constraints on the Galileon model at the background level by using the combined datasets of
Supernovae Ia, CMB shift parameters, and BAO
We also include the cosmic curvature K.
For the tracker the Hubble parameter is known in terms of the redshift z, as
$\left(\frac{H(z)}{H_0}\right)^2 = \frac{1}{2}\Omega_K^{(0)}(1+z)^2 + \frac{1}{2}\Omega_m^{(0)}(1+z)^3 + \frac{1}{2}\Omega_r^{(0)}(1+z)^4 + \sqrt{1-\Omega_m^{(0)}-\Omega_r^{(0)}-\Omega_K^{(0)}+\frac{(1+z)^4}{4}\left[\Omega_K^{(0)}+\Omega_m^{(0)}(1+z)+\Omega_r^{(0)}(1+z)^2\right]^2}$
where $\Omega_K^{(0)} = -K/(a_0 H_0)^2$
For general solutions that approach the tracker at late times we need to solve the background equations numerically.
S. Tsujikawa"Cosmology of a Galileon Field" presentation slide

Ref.

[1]Shiniji Tsujikawa, Antonio De Felice and Ryoutaro "Cosmology of a Galileon Field"*Phys. Rev. Lett.***105**, 111301 (2010)
[2]Antonio De Felice and Shinji Tsujikawa, "Cosmology of a Covariant Galileon Field" Presentation Slides (2010)
[3]Antonio De Felice, Shinji Tsujikawa, "Generalized Galileon cosmology" arXiv:1008.4236 (2010)
[4]Savvas Nesseris, Antonio De Felice, Shinji Tsujikawa, "Observational constraints on Galileon cosmology" arXiv:1010.0407 (2010)

Ref.

[1]Shiniji Tsujikawa, Antonio De Felice and Ryoutaro "Cosmology of a Galileon Field"*Phys. Rev. Lett.***105**, 111301 (2010)
[2]Antonio De Felice and Shinji Tsujikawa, "Cosmology of a Covariant Galileon Field" Presentation Slides (2010)
[3]Antonio De Felice, Shinji Tsujikawa, "Generalized Galileon cosmology" arXiv:1008.4236 (2010)
[4]Savvas Nesseris, Antonio De Felice, Shinji Tsujikawa, "Observational constraints on Galileon cosmology" arXiv:1010.0407 (2010)

◂ Comparison Table

Setup of the Model

(4) Gauss-Bonnet Gravity

(0) Mathematical Tool
(1) Scalar-Tensor Theories
(2) f(R) Theories
(3) Galileon Gravity
(4) Gauss-Bonnet Gravity
(5) Chern-Simons(CS) Theories
(6) Hořava-Lifshitz(HL) Gravity
(7) Entropic Forces

(4) Gauss-Bonnet Gravity

(4) Gauss-Bonnet Gravity

In general relativity, Einstein-Gauss-Bonnet gravity is a modification of the Einstein-Hilbert action to include the Gauss-Bonnet term $\mathcal{G} := R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, then the EH-action will rewrite like:

(4) Gauss-Bonnet Gravity

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This term is only nontrivial in 4+1D or greater, and as such, only applies to extra dimensional models. In 3+1D and lower, it reduces to a topological surface term. Despite being quadratic in the Riemann tensor (and Ricci tensor), terms containing more than 2 partial derivatives of the metric cancel out, making the Euler-Lagrange equations second order quasilinear partial differential equations in the metric.

(4) Gauss-Bonnet Gravity

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Consequently, there are no additional dynamical degrees of freedom, as in say f(R) gravity.

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More generally, we may consider

$\int d^D\sqrt{-g}f(\mathcal{G})$

term for some function f. Nonlinearities in f render this coupling nontrivial even in 3+1D. However, fourth order terms reappear with the nonlinearities.

(4) Gauss-Bonnet Gravity

Ref.

Lovelock, David (1971), "The Einstein tensor and its generalizations", *J. Math. Phys.* **12**(3), 498

(4) Gauss-Bonnet Gravity

Ref.

Lovelock, David (1971), ”The Einstein tensor and its generalizations”, *J. Math. Phys.* **12**(3), 498

Comparison Table

Setup of the Model

(5) Chern-Simons(CS) Theories

(0) Mathematical Tool
(1) Scalar-Tensor Theories
(2) f(R) Theories
(3) Galileon Gravity
(4) Gauss-Bonnet Gravity
(5) Chern-Simons(CS) Theories
(6) Hořava-Lifshitz(HL) Gravity
(7) Entropic Forces

(5) Chern-Simons(CS) Theories

The low-energy limit of string theory contains an anomaly-canceling correction to the Einstein-Hilbert action, which defines an effective theory: Chern-Simons (CS) modified gravity.

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The CS correction consists of the product of a scalar field with the Pontryagin density, where the former can be treated as a background field (nondynamical formulation) or as an evolving field (dynamical formulation).

(5) Chern-Simons(CS) Theories

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1. 3+1,non-dynamical CS theory

(5) Chern-Simons(CS) Theories

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The CS correction consists of the product of a scalar field with the Pontryagin density, where the former can be treated as a background field (nondynamical formulation) or as an evolving field (dynamical formulation).

1. 3+1,non-dynamical CS theory
2. 3+1,dynamical CS theory

(5) Chern-Simons(CS) Theories

Dynamical Action

$$\mathcal{S} = \frac{1}{2\kappa^2}\int_{\mathcal{V}} d^4x\,\sqrt{-g}R - \frac{\pm\alpha}{4}\int_{\mathcal{V}} d^4x\,\sqrt{-g}\,(\vartheta\,{}^*RR) + \beta\,\frac{-1}{2}\int_{\mathcal{V}} d^4x\,\sqrt{-g}[\,g^{ab}(\nabla_a\vartheta)(\nabla_b\vartheta) + 2V(\vartheta)\,]$$

When discussing Solar System tests of CS modified gravity we choose $\alpha = \frac{-\ell}{3}$ and $\beta = -1$.

Notation

Only in this section latin letters at the beginning of the alphabet a, b, . . . , h correspond to spacetime indices, while those at the end of the alphabet i, j, . . . , z stand for spatial indices only. R is the Ricci scalar, and the integrals are volume ones carried out everywhere on the manifold $\mathcal{V}$.

The quantity ${}^*RR$ is the Pontryagin density, defined via

$${}^*RR := R\widetilde{R} = {}^*R^a{}_b{}^{cd}R^b{}_{acd}$$

where the dual Riemann-tensor is given by

$${}^*R^a{}_b{}^{cd} := \frac{1}{2}\epsilon^{cdef}R^a{}_{bef}$$

(5) Chern-Simons(CS) Theories

Ref. Stephon Alexander and Nicólas Yunes, ArXiv:0907.2562

Notation

with ϵ^{cdef} the 4-dimensional Levi-Civita tensor.
The quantity ϑ is the so-called CS coupling field, which is not a constant, but a function of spacetime, thus serving as a deformation function.

Notation

Formally, if ϑ = const. CS modified gravity reduces identically to GR. This is because the Pontryagin term can be expressed as the divergence
$\nabla_a K^a = \frac{1}{2} {}^*RR$
of the Chern-Simons topological current
$K^a := \epsilon^{cdef} \Gamma^n_{bm} (\partial_c \Gamma^m_{dn} + \frac{2}{3} \Gamma^m_{cl} \Gamma^l_{dn})$
where Γ here is the Christoffel connection.

(5) Chern-Simons(CS) Theories

Ref. Stephon Alexander and Nicólas Yunes, ArXiv:0907.2562

Reduced to GR

One can now integrate S_{CS} by parts to obtain
$S_{CS} = \alpha(\vartheta K^a)\,|_{\partial\mathcal{V}} - \frac{\alpha}{2}\int_{\mathcal{V}} d^4x\sqrt{-g}(\nabla_a\vartheta)K^a)$
where the first term is usually discarded since it is evaluated on the boundary of the manifold.

The second term clearly depends on the covariant derivative of ϑ, which vanishes if ϑ = const. and, in that case, CS modified gravity reduces to GR.

(5) Chern-Simons(CS) Theories

Ref. Stephon Alexander and Nicólas Yunes, ArXiv:0907.2562

Away from GR

For any finite, yet arbitrarily small $\nabla_a\vartheta$, CS modified gravity becomes substantially different from GR. The quantity $\nabla_a\vartheta$ can be thought of as an embedding coordinate, because it embeds a generalization of the standard 3-dimensional CS theory into a 4-dimensional spacetime. In this sense, $\nabla_a\vartheta$ and $\nabla_a\nabla_b\vartheta$ act as deformation parameters in the phase space of all theories. One can then picture GR as a stable fixed point in this phase space. Away from this "saddle point," CS modified gravity induces corrections to the Einstein equations that are proportional to the steepness of the ϑ deformation parameter.

Aspects of CS gravity

Spinning black holes in the slow-rotation approximation

Many solutions of general relativity persist in the modified theory; a notable exception is the Kerr metric, which has sparked a search for rotating black hole solutions.
Ref. Nicola's Yunes and Frans Pretorius, *Phys. Rev. D*,**79** 084043(2009)

Aspects of CS gravity

In String Theory

”The absence of a CS term in the action leads to the Green-Schwarz anomaly, which requires cancellation to preserve unitarity and quantum consistency. In most perturbative string theories (e.g. type IIB, I, heterotic) with four-dimensional compactifications, the Green-Schwarz mechanism requires the inclusion of a CS term.

Ref. Stephon Alexander and Nicola's Yunes, *Phys. Rev. D*,**77**, 124040(2008)

Aspects of CS gravity

In String Theory

In fact, this term is induced in all string theories due to duality symmetries in the presence of Ramond-Ramond scalars or D-instanton charges. Even in heteroticM theory the CS term is required through the use of an anomaly inflow."

Aspects of CS gravity

In Loop Quantum Gravity

"CS term arises as a natural extension to the Hamiltonian constraint. In particular, the CS term renders a candidate holomorphic ground state wavefunction invariant under large gauge transformations of the Ashtekar connection variables. The CS correction, is also related to the Immirzi parameter of loop quantum gravity, which determines the spectrum of quantum geometrical operators."

Aspects of CS gravity

parity-violating

”CS modified gravity proposes an extension to GR by adding a parity-violating, Chern-Pontryagin term to the Einstein-Hilbert action, multiplied by a spacetimedependent coupling scalar. This theory modifies the GR field equations by adding a new cottonlike C-tensor, which is composed of derivatives of the Ricci tensor and the dual to the Riemann. Additionally, the equations of motion for the scalar field provide a new Pontryagin constraint that preserves diffeomorphism invariance.

Aspects of CS gravity

parity-violating

The structure of the C-tensor allows the modified theory to preserve some of the classical solutions of GR, such as the Schwarzschild, the Friedmann-Robertson-Walker, and the gravitational wave line elements."

Aspects of CS gravity

parity-violating

”Although some classic GR solutions are preserved in CS modified gravity, parity violation is inherent in the modified theory, leading to possibly observable effects. One such effect is amplitude birefringence, which leads to a distinct imprint that could be detectable through gravitational wave observations. ”

Aspects of CS gravity

parity-violating

”Birefringent gravitational waves have actually been successfully employed to propose an explanation to the leptogenesis problem and could also leave an imprint in the cosmic-microwave background. Another consequence of CS modified gravity is modified precession, which has been studied in the far-field limit, leading to a weak bound on the CS scalar with LAGEOS.

Aspects of CS gravity

parity-violating

Recent investigations have also concentrated on spinning black hole solutions, as well as black hole perturbations, both of which have been seen to be corrected in CS modified gravity."

Ref. Stephon Alexander and Nicola's Yunes, *Phys. Rev. D*,**77**, 124040(2008)

Aspects of CS gravity

parity-violating

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Ref. Stephon Alexander and Nicola's Yunes, *Phys. Rev. D*,**77**, 124040(2008) Comparison Table

Setup of the Model
(6) Hořava-Lifshitz(HL) Gravity

- (0) Mathematical Tool
- (1) Scalar-Tensor Theories
- (2) f(R) Theories
- (3) Galileon Gravity
- (4) Gauss-Bonnet Gravity
- (5) Chern-Simons(CS) Theories
- (6) Hořava-Lifshitz(HL) Gravity
- (7) Entropic Forces

Hořava-Lifshitz gravity

1. IR fixed point: General Relativity
2. Hořava-Lifshitz gravity (or Hořava gravity) is a theory of quantum gravity proposed by Petr Hořava in 2009.
3. It solves the problem of different concepts of time in quantum field theory and general relativity by treating the quantum concept as the more fundamental so that space and time are not equivalent (anisotropic).
4. The relativistic concept of time with its Lorentz invariance emerges at large distances.
5. The theory relies on the theory of foliations to produce its causal structure.

Hořava-Lifshitz gravity

1. It is a possible UV completion of general relativity.
2. The novelty of this approach, compared to previous approaches to quantum gravity such as Loop quantum gravity, is that it uses concepts from condensed matter physics such as quantum critical phenomena.
3. Hořva initial formulation was found to have side-effects such as predicting very different results for a spherical Sun compared to a slightly non-spherical Sun, so others have modified the theory. Inconsistencies remain.
4. It is related to topologically massive gravity and the Cotton tensor.

Hořava-Lifshitz gravity— Extrinsic Curvature

Extrinsic Curvature

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$$

Ref.

Kiritsis, Kofinas, ”Interacting String Multi-Verses And Holographic Instabilities Of Massive Gravity” *Nucl. Phys. B*,**812**:488-524 (2009) ,arXiv:0808.3410

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◂ Comparison Table

Setup of the Model

(7) Entropic Forces

- (0) Mathematical Tool
- (1) Scalar-Tensor Theories
- (2) f(R) Theories
- (3) Galileon Gravity
- (4) Gauss-Bonnet Gravity
- (5) Chern-Simons(CS) Theories
- (6) Hořava-Lifshitz(HL) Gravity
- (7) Entropic Forces

(7) Entropic Forces

Intro.

Entropic gravity is a hypothesis in modern physics that describes gravity as an entropic force; that it is not a fundamental interaction mediated by a particle, but a probabilistic consequence of physical systems' tendency to increase their entropy.

Ref. en.wikipedia.org

(7) Entropic Forces

Origin

Entropic gravity has a history that goes back to research on black hole thermodynamics by Bekenstein and Hawking in the mid-1970s. These studies suggest a deep connection between gravity and thermodynamics, which describes the behavior of heat and gases. In 1995, Jacobson demonstrated that the Einstein equations describing relativistic gravitation can be derived by combining general thermodynamic considerations with the equivalence principle.[1] Subsequently, other physicists began to explore links between gravity and entropy.[2]

(7) Entropic Forces

Ref.

[1]Jacobson, Theodore (4 April 1995). ”Thermodynamics of Spacetime: The Einstein Equation of State”. Phys. Rev. Lett.75.1260. arxiv:gr-qc 9504004. Retrieved 6 September 2010.

[2]Padmanabhan, Thanu (26 November 2009). ”Thermodynamical Aspects of Gravity: New insights”. arxiv:0911.5004. Retrieved 6 September 2010.

(7) Entropic Forces

Erik Verlinde's theory

In 2009, Erik Verlinde disclosed a conceptual theory that describes gravity as an entropic force.[3] On January 6, 2010 he published a preprint of a 29 page paper titled "On the Origin of Gravity and the Laws of Newton".[4] Reversing the logic of over 300 years, it argued that gravity is a consequence of the laws of thermodynamics. This theory combines the thermodynamic approach to gravity with Gerardus 't Hooft's holographic principle.

(7) Entropic Forces

Erik Verlinde's theory

If proven correct, this implies gravity is not a fundamental interaction, but an emergent phenomenon which arises from the statistical behavior of microscopic degrees of freedom encoded on a holographic screen. The paper drew a variety of responses from the scientific community. Andrew Strominger, a string theorist at Harvard said "Some people have said it can't be right, others that it's right and we already knew it —that it's right and profound, right and trivial."[5]

(7) Entropic Forces

Erik Verlinde's theory

Verlinde's article also attracted a large amount of media exposure,[6][7] and led to immediate follow-up work in cosmology,[8][9] the dark energy hypothesis,[10] cosmological acceleration,[11][12] cosmological inflation,[13] and loop quantum gravity.[14] Also, a specific microscopic model has been proposed that indeed leads to entropic gravity emerging at large scales.[15]

(7) Entropic Forces

Ref.

[3]van Calmthout, Martijn ”Is Einstein een beetje achterhaald?” (in Dutch). de Volkskrant. (2010)
[4] Verlinde, Eric . ”Title: On the Origin of Gravity and the Laws of Newton”. ArXiv:1001.0785. (2010).
[5] Overbye, Dennis . ”A Scientist Takes On Gravity”. The New York Times.
[6] The entropy force: a new direction for gravity, New Scientist, (2010), issue 2744.

(7) Entropic Forces

Ref.

[5]Gravity is an entropic form of holographic information, Wired Magazine,(2010)
[6]Equipartition of energy and the first law of thermodynamics at the apparent horizon, Fu-Wen Shu, Yungui Gong, arXiv:1001.3237v1(2010)
[7]Friedmann equations from entropic force, Rong-Gen Cai, Li-Ming Cao, Nobuyoshi Ohta arXiv:1001.3470(2010)
[8]It from Bit: How to get rid of dark energy, Johannes Koelman, (2010)

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Ref.

[9]Entropic Accelerating Universe, Damien Easson, Paul Frampton, George Smoot, arXiv:1002.4278 (2010)
[10]Entropic cosmology: a unified model of inflation and late-time acceleration, Yi-Fu Cai, Jie Liu, Hong Li,arXiv:1003.4526 (2010)
[11]Towards a holographic description of inflation and generation of fluctuations from thermodynamics, Yi Wang, arXiv:1001.4786 (2010)
[12]Newtonian gravity in loop quantum gravity, Lee Smolin, arXiv:1001.3668v1(2010)

[13] Notes concerning "On the origin of gravity and the laws of Newton" by E. Verlinde, Jarmo Makela, arXiv:1001.3808(2010)

Comparison Table

Outlook and Future Works

OPEN QUESTION 1 Physical Effects

What's the physical effects about each kind of the correcting terms in each ETG/MG models?
Cosmological constraints may give some clues.

OPEN QUESTION 2 JF versus EF

Are ETG/MGs physically equivalent in JF and EF?
Cosmological constraints may give some clues.

Related Ref.

V. Faraoni and E. Gunzig, "Einstein Frame or Jordan Frame?"Int. J. Theor. Phys. 38 (1999), p. 217

OPEN QUESTION 3 Appy CF to CS and HL

Since quintessence field can be consider as a coupling scalar field in dynamical CS and HL gravity, one can try to appy conformal transformation and transform the CS-action and HL-action into Einstein frame.

OPEN QUESTION 4 Uniquness of Action

Why so many types of action? And, what's the relation between each ETG/MG models themselves?
Is the action of gravitaional field unique?
Cosmological constraints may give some clues.

Related Research about f(R)-theories and Hořava-Lifshitz(HL) Gravity

Quick Overview

[1]Kluson, J., ”Horava-Lifshitz f(R) gravity”, J. High Energy Phys., 2009(11), 078, (2009).
[2]Kluson, J., ”New Models of f(R) Theories of Gravity”, *Phys. Rev.*, **81**, 064028, (2010).,arXiv:0910.5852.

OPEN QUESTION 5 Hamiltonian formalism and Quantization

How about Hamiltonian formalism?
What is physics that we can find through this formalism?
Could we obtain some clues which direct to the quantization(or quantum gravity)?

OPEN QUESTION 6 Bridge to Quantum Gravity

How to find the bridge from ETG/MGs to quantum gravity or gauge gravity or more fundamental theory(SUSY, String theory, Loop Quantum Gravity and Horava Gravity...etc.)?