Controversial Issue: Physical Equivalence of JF and EF

William Chuang

December 21, 2010

Outline

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- Conformal Transformation
- Controversial Issue: Physical Equivalence of JF and EF
- Outlook

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Conformal Transformation —Applications

- 1984 Wald
  GR: Asymptotic flatness and in the initial value formulation.
- Fermat Principal
  Perlick 1990, Schneider, Ehlers and Falco 1992
- Gravitational Lensing in the (conformally flat)
  Friedmann-Lemaitre-Robertson-Walker Universe
  Perlick 1990, Schneider, Ehlers and Falco 1992

Conformal Transformation —Applications

- Wave Equations
  Sonego and Faraoni 1992
  Nonan 1995
- Optical Geometry near black hole horizons
  Abramowicz, Carter and Lasota 1988
  Sonego and Massar 1997
  Abramowicz et al. 1997
- Exact Solutions
  Van der Bergh 1986, 1988

Conformal Transformation —Applications

- Quantum Field Theory in Curved Spaces
  Birell and Davies 1982
- Statistical Mechanics and String Theories
  Dita and Georgescu 1989

Conformal Transformation —Intro.

which is often used as a mathematical tool to map the equations of motion of physical systems into mathematically equivalent sets of equations that are more easily solved and computationnally more convenient to study.

Conformal Transformation —Intro.

which is mainly used in three different areas of gravitational physics:

1. Alternative (including nonlinear) theories of gravity

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1. Alternative (including nonlinear) theories of gravity
2. Unified theories in multi-dimensional spaces
3. Scalar fields non-minimally coupled to gravity

Conformal Transformation —Intro.

more precisely description

A conformal transformation is essentially a local change of scale.

Conformal Transformation —Intro.

more precisely description

Since the distance are measured by the metric, hence the metric is multiplied by a spacetime-dependent (nonvanishing) function.

$\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}$

i.e. $\tilde{d}^2 s = \omega^2(x) d^2 s$

for some nonvanishing function $\omega(x)$ x is used to denote the collection of spacetime coordinate x^μ.

Note that

$g_{\mu\nu} = \omega^{-2}(x)\tilde{g}_{\mu\nu}$ is trivial.

Conformal Transformation —Intro.

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with this sense we have two way to appy

Applications

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Applications

1. To change dynamical variables in a scalar-tensor theories.

Conformal Transformation —Intro.

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Applications

1. To change dynamical variables in a scalar-tensor theories.
2. Remap spacetimes into convenient conformal (Penrose)diagram.

Conformal Transformation —A Critical Fact

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Null curves are left invariant by conformal transformations.

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Theorem

A curve $x^\mu(x)$ is null iff. its tangent vector $\frac{dx^\mu}{d\lambda}$ is null.

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Extension—Property

By the theorem above we can show that ”Conformal transformation leave light cones invariant.”

Conformal Transformation —A Critical Fact

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Proof of Extended Property

1. By the Thm.

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2. $g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0$
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   $\tilde{g}_{\mu\nu} = \omega^2(x)g_{\mu\nu}$
   substisude into above equation

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Anything that is function of $g_{\mu\nu}$ can be equally well thought of as a function of $\tilde{g}_{\mu\nu}$ and $\omega(x)$.
Then it can be called that the quantities are expressed in the conformal frame.

Conformal Transformation

Christoffel Symbols

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \frac{1}{\omega}(\delta^{\rho}_{\mu}\nabla_{\nu}\omega + \delta^{\rho}_{\nu}\nabla_{\mu}\omega - g_{\mu\nu}g^{\rho\lambda}\nabla_{\lambda}\omega)$$

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$$= \Gamma^{\rho}_{\mu\nu} + C^{\rho}_{\mu\nu}$$

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$$= \Gamma^{\rho}_{\mu\nu} + C^{\rho}_{\mu\nu}$$

where $C^{\rho}_{\mu\nu} := \frac{1}{\omega}(\delta^{\rho}_{\mu}\nabla_{\nu}\omega + \delta^{\rho}_{\nu}\nabla_{\mu}\omega - g_{\mu\nu}g^{\rho\lambda}\nabla_{\lambda}\omega)$

Conformal Transformation

Conformal Transformation

The first covariant derivative of a scalar field ϕ

$\widetilde{\nabla}_\mu \phi$

Conformal Transformation

The first covariant derivative of a scalar field ϕ

$$\widetilde{\nabla}_{\mu}\phi = \nabla_{\mu}\phi$$

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$$\widetilde{\nabla}_\mu \phi = \nabla_\mu \phi = \partial_\mu \phi$$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

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The second covariant derivative of a scalar field ϕ

In Conformal Frame

$\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \phi$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

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$$\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi$$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

In Conformal Frame

$$\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\nabla_\alpha \omega)(\nabla_\beta \phi)$$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

In Conformal Frame

$$\widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\nabla_\alpha \omega)(\nabla_\beta \phi)$$

In Original Frame

$$\nabla_\mu \nabla_\nu \phi$$

Conformal Transformation

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In Original Frame

$$\nabla_\mu \nabla_\nu \phi = \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi$$

Conformal Transformation

The second covariant derivative of a scalar field ϕ

In Conformal Frame

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\nabla_\alpha \omega)(\nabla_\beta \phi)$$

In Original Frame

$$\nabla_\mu \nabla_\nu \phi = \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + (\delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu - g_{\mu\nu} g^{\alpha\beta}) \omega^{-1} (\tilde{\nabla}_\alpha \omega)(\tilde{\nabla}_\beta \phi)$$

Conformal Transformation
D' Alembertian

In Conformal Frame

$\widetilde{\Box}\phi$

Conformal Transformation
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$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi$$

Conformal Transformation
D' Alembertian

In Conformal Frame

$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\omega)(\nabla_\beta\phi)$$

Conformal Transformation
D' Alembertian

In Conformal Frame

$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\omega)(\nabla_\beta\phi)$$

In Original Frame

$$\Box\phi$$

Conformal Transformation

D' Alembertian

In Conformal Frame

$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_{\alpha}\omega)(\nabla_{\beta}\phi)$$

In Original Frame

$$\Box\phi = \omega^{2}\widetilde{\Box}\phi$$

Conformal Transformation

D' Alembertian

In Conformal Frame

$$\widetilde{\Box}\phi = \omega^{-2}\Box\phi + (n-2)g^{\alpha\beta}\omega^{-3}(\nabla_\alpha\omega)(\nabla_\beta\phi)$$

In Original Frame

$$\Box\phi = \omega^{2}\widetilde{\Box}\phi - (n-2)\tilde{g}^{\alpha\beta}\omega(\widetilde{\nabla}_\alpha\omega)(\widetilde{\nabla}_\beta\omega)$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$\widetilde{R}$

Conformal Transformation
Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega)$$

Conformal Transformation
Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega)$$
$$- (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

In Original Frame

R

Conformal Transformation
Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

In Original Frame

$$R = \omega^2 \widetilde{R}$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

In Original Frame

$$R = \omega^2 \widetilde{R} + 2(n-1) \tilde{g}^{\alpha\beta} \omega (\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \omega)$$

Conformal Transformation

Ricci Scalar

In Conformal Frame

$$\widetilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$

In Original Frame

$$R = \omega^2 \widetilde{R} + 2(n-1) \tilde{g}^{\alpha\beta} \omega (\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta \omega) - n(n-1) \tilde{g}^{\alpha\beta} (\widetilde{\nabla}_\alpha \omega)(\widetilde{\nabla}_\beta \omega)$$

Mathematical Tool: Conformal Transformation

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Weinberg, S ”Gravitation and Cosmology” John Wiley, New York N.Y. (1972)

Mathematical Tool: Conformal Transformation

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Physical non-equivalence of the Jordan and Einstein frames

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ABSTRACT

We show, considering a specific $f(R)$-gravity model, that the Jordan frame and the Einstein frame could be physically non-equivalent, although they are connected by a conformal transformation which yields a mathematical equivalence. Calculations are performed analytically and this non-equivalence is shown in an unambiguous way. However this statement strictly depends on the considered physical quantities that have to be carefully selected.

$$\mathcal{A} = \int d^4x \sqrt{-g} f(R) + \mathcal{A}_m, \tag{1}$$

where $f(R)$ is a generic function of the Ricci scalar R and $\mathcal{A}_m$ is the action of a perfect fluid minimally coupled with gravity. Obviously assuming $f(R) = R$ the standard Einstein theory is recovered. Varying with respect to $g_{\mu\nu}$, we get the field equations

$$G_{\mu\nu} = T^{curv}_{\mu\nu} + \frac{T^m_{\mu\nu}}{2f'(R)}, \tag{2}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{3}$$

and $T^{curv}_{\mu\nu}$ is an effective stress–energy tensor constructed by curvature terms in the following way

$$T^{curv}_{\mu\nu} = \frac{1}{f'(R)}\left\{\frac{1}{2}g_{\mu\nu}\left[f(R) - Rf'(R)\right] + f'(R)_{;\mu\nu} - g_{\mu\nu}f'(R)^{;\alpha}_{;\alpha}\right\}. \quad (4)$$

This tensor is zero for $f(R) = R$. The prime indicates derivatives with respect to R.

In a Friedmann–Robertson–Walker (FRW) metric, taking into account a dust-matter perfect fluid, a point-like Lagrangian can be obtained

$$\mathcal{L} = a^3\left[f(R) - f'(R)R\right] + 6a^2 f''(R)\dot{R}\dot{a} + 6f'(R)a\dot{a}^2 - 6kf'(R)a + D, \quad (5)$$

where D represents the standard amount of dust fluid, such that $\rho = D/a^3$ [31]. The energy function $E_{\mathcal{L}}$, corresponding to the hat the PPN parameters of ied. In a recent paper [28], sed considering that fourth y equivalent to the O'Han- l case of scalar-tensor grav- ction potential and that, in non-standard behavior that PPN limit of GR. The result from the one of Brans–Dicke hat it is misleading to con- ory with $\omega = 0$ in order to es of $f(R)$-gravity. In other in indication of the fact that ld *not* be physically equiva- tement has to be supported observable physical quanti- it of the frames or, at least, into the frames well estab-

ve that the physical non- stein frame could be exactly model and selecting physi- n, we will take into account mpare analytically the two es.

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where D represents the standard amount of dust fluid, such that $\rho = D/a^3$ [31]. The energy function $E_{\mathcal{L}}$, corresponding to the $\{0,0\}$-Einstein equation, is

$$E_{\mathcal{L}} = 6f''(R)a^2\dot{a}\dot{R} + 6f'(R)a\dot{a}^2 - a^3\left[f(R) - f'(R)R\right] + 6kf'(R)a - D = 0. \quad (6)$$

The equations of motion for a and R are respectively

$$f''(R)\left[R + 6H^2 + 6\frac{\ddot{a}}{a} + 6\frac{k}{a^2}\right] = 0, \quad (7)$$

$$6f'''(R)\dot{R}^2 + 6f''(R)\ddot{R} + 6f'(R)H^2 + 12f'(R)\frac{\ddot{a}}{a} = 3\left[f(R) - f'(R)R\right] - 12f''(R)H\dot{R} - 6f'(R)\frac{k}{a^2}, \quad (8)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. Eq. (7) ensures the consistency, since R coincides with the definition of the Ricci scalar in

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$$E_{\mathcal{L}} = -\frac{9}{2}a^2|R|^{-1/2}\dot{R}\dot{a} + 9|R|^{1/2}a\dot{a}^2 - \frac{a^3}{2}|R|^{3/2} + 9k|R|^{1/2}a - D = 0. \tag{10}$$

Referring to [30], it is possible to show that such a model has a Noether symmetry that allows to find out an exact solution for Eqs. (6), (7) and (8) for this particular $f(R)$, that is

$$a(t) = \sqrt{a_4t^4 + a_3t^3 + a_2t^2 + a_1t} \tag{11}$$

with

$$a_4 = \frac{\Sigma_1^2}{144}, \qquad a_3 = \frac{\Sigma_1\Sigma_0}{36}, \qquad a_2 = \frac{\Sigma_0^2}{24} - k,$$
$$a_1 = \frac{\Sigma_0^3}{36\Sigma_1} - 2k\frac{\Sigma_0}{\Sigma_1} + \frac{4D}{9\Sigma_1}, \tag{12}$$

where k is the spatial curvature, Σ_1 the Noether charge and Σ_0 the integration constant.

In order to fix the coefficients $a_i's$, we have to consider time units in which the current time is $t_0 = 1$. However, one can construct the dimensionless quantity $H_0t_0 \sim 0.93$ which has to remain constant. Therefore the Hubble parameter results of order one (we choose $H_0 = 1$ for simplicity). The current deceleration parameter can also be fixed taking $q_0 = -0.4$, which could describe a reasonable current acceleration. Finally, a unit value for the present scale factor value is considered. This assumption can be always done if no restriction on the value of k is imposed. In order to fix the remaining free parameters, we consider $a_4 = 0.106$, which leads $\Omega_{m0} = 0.0418032$ (with $\Omega_m = \rho/[6H^2 f'(R)]$), very close to the expected content of baryonic matter. With these assumptions, the scale factor is

which, by defining a auxiliary scalar field φ in the following way,

$$\varphi(R) = \sqrt{\frac{3}{2}}\ln\left(3|R|^{1/2}\right), \tag{16}$$

can be written as

$$\mathcal{A}_G = \int d^4x\sqrt{-g}\left[-\frac{|R|}{2}e^{\sqrt{2/3}\varphi} + \frac{1}{54}e^{3\sqrt{2/3}\varphi}\right]. \tag{17}$$

The new field φ does not introduce any physical new feature, since it is only a way to recast the further gravitational degrees of freedom related to $f(R)$-gravity. In fact, it can be seen that this is the case, since the φ-field equation obtained from Eq. (17) produces only Eq. (16). If we perform a conformal transformation by the conformal parameter

$$b(t) = \exp\left(\frac{\varphi}{2}\sqrt{\frac{2}{3}}\right), \tag{18}$$

which is a function of the time t since $\varphi(R(t)) = \varphi(t)$, the resulting action is the Hilbert–Einstein action with a scalar field $\varphi(t)$

$$\mathcal{A}_G = \int d^4x\sqrt{-\bar{g}}\left[-\frac{|\bar{R}|}{2} - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\right], \tag{19}$$

where $\|\bar{g}_{\mu\nu}\| = b(t)^2\,\mathrm{diag}(-1, a(t)^2, a(t)^2, a(t)^2)$, $\bar{R}$ is the Ricci scalar of the metric $\bar{g}_{\mu\nu}$ and $V(\varphi) = \exp[\sqrt{2/3}\varphi]/54$. If we define a new time variable τ, in such a way that $d\tau = b(t)dt$, we recover a FRW metric $\tilde{g}_{\mu\nu}$, but now with a scale factor $a_E(\tau) = b(\tau)a(\tau)$

$$\mathcal{A}_G = \int d^4\tilde{x}\sqrt{-\tilde{g}}\left[-\frac{|\tilde{R}|}{2} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\tilde{\varphi}\tilde{\partial}_\nu\tilde{\varphi} + \tilde{V}(\tilde{\varphi})\right]. \tag{20}$$

$$a(t) = \sqrt{\frac{t}{5}\left[2 + 0.53(t-1)^3 + t + 2t^2\right]} \tag{13}$$

and the Ricci scalar

$$R(t) = \frac{9(41 + 212t)^2}{212t(147 + 259t + 41t^2 + 53t^3)}. \tag{14}$$

This model describes a spatially open Universe, $k \simeq -0.5$. We have to note that the measurable quantity is not this parameter but $\Omega_{k0} \simeq 0.02$ which is very small. Moreover, since the requirement $\Omega_k \simeq 0$ is derived by the spectrum of the CMBR data, and these data strongly depend on the standard ΛCDM model, we cannot conclude that this feature is needed in our $f(R)$-model.

In fact, this solution, in principle, seems to reproduce satisfactorily observational data, out from the trivial fulfillment of the *a priori* fixed. In particular, the scale factor (13) is able to emulate a dust dominated epoch necessary for the structure formation, with only a difference with respect the standard $a_F \sim t^{2/3}$ of the 3% in the range $2 \leqslant z \leqslant 4$, and the distance modulus derived by this model is also able to reproduce the SNeIa data [30].[2]

3. Conformal transformation

Let us consider now the gravitational part of our action, i.e.

$$\mathcal{A}_G = -\int d^4x\sqrt{-g}|R|^{3/2}, \tag{15}$$

[2] This choice of the parameters is interesting because it produces results which turn out to be reasonably good at least from the point of view of observational tests. However, the following comparison with the Einstein frame is not dependent on this choice.

$\tilde{R}$ is the Ricci scalar of the metric $\tilde{g}_{\mu\nu}$, $\tilde{R}(\tau) = \bar{R}(t(\tau))$, $\tilde{\varphi}(\tau) = \varphi(t(\tau))$ and $\tilde{V}(\tilde{\varphi}) = V(\varphi)$. Taking also into account the mentioned transformations in the matter component, we obtain the total action in the Einstein frame and the point-like FRW Lagrangian

$$\mathcal{L} = 3a_E(\partial_\tau a_E)^2 - 3ka_E - \frac{a_E^3}{2}(\partial_\tau\tilde{\varphi})^2 + a_E^3\tilde{V}(\tilde{\varphi}) + e^{-\tilde{\varphi}/\sqrt{6}}\tilde{\rho}_m, \tag{21}$$

where $\tilde{\rho}_m = D/a_E^3$. Such a Lagrangian shows a coupling between the matter term and the scalar field, which will produce the non-conservation of both fluids individually.

The Einstein equations yield

$$\tilde{G}_{\mu\nu} = \tilde{T}^{\tilde{\varphi}}_{\mu\nu} + \tilde{T}^{m}_{\mu\nu} + \tilde{T}^{int}_{\mu\nu}, \tag{22}$$

where

$$\tilde{T}^{\tilde{\varphi}}_{\mu\nu} = \tilde{\partial}_\mu\tilde{\varphi}\tilde{\partial}_\nu\tilde{\varphi} - \frac{1}{2}\tilde{\partial}_\alpha\tilde{\varphi}\tilde{\partial}^\alpha\tilde{\varphi}\tilde{g}_{\mu\nu} + \tilde{V}(\tilde{\varphi})\tilde{g}_{\mu\nu}, \tag{23}$$

$$\tilde{T}^{m}_{\mu\nu} = \mathrm{diag}(\tilde{\rho}_m, 0, 0, 0), \tag{24}$$

and

$$\tilde{T}^{int}_{\mu\nu} = \left(e^{-\tilde{\varphi}/\sqrt{6}} - 1\right)\mathrm{diag}(\tilde{\rho}_m, 0, 0, 0). \tag{25}$$

It should be noted that, whereas $\tilde{T}^m_{\mu\nu}$ is conserved $\tilde{T}^{\tilde{\varphi}}_{\mu\nu}$ and $\tilde{T}^{int}_{\mu\nu}$ do not fulfill any conservation law separately, but $(\tilde{T}^{\tilde{\varphi}}_{\mu\nu} + \tilde{T}^{int}_{\mu\nu})^{;\mu} = 0$. This result has to be taken into account in order to compare results in Jordan and Einstein frames.

Re-examine of This Paper

4. Jordan frame versus Einstein frame

In the previous section, we have shown how to perform a conformal transformation of $f(R)$-gravity to obtain GR with a dynamical scalar field, being therefore both frames mathematically equivalent. However, this mathematical equivalence does not necessary ensure the physically equivalence of both frames. In fact, whereas, in the Jordan frame, the matter term is not coupled to any field or to gravity, in the Einstein frame there is a coupling between the matter and the scalar field, appearing as an interaction term in the Einstein equations (22). This fact is crucial in comparing the physics in the two systems.

In order to show that the two frames could be physically equivalent, we have to compare the physical quantities of the mentioned two frames. This is a delicate issue since the selection of such quantities should be unambiguous.

Through the definition of the conformal factor, Eq. (18), and Eqs. (14) and (16), one finds the explicit form of this parameter in terms of t

$$b(t) = \frac{3\sqrt{41+212t}}{\sqrt{106}(147t+259t^2+41t^3+53t^4)^{1/4}}, \qquad (26)$$

with t the cosmic time in the Jordan frame, which is related to the cosmic time in the Einstein frame

$$\tau = \int b(t)\,dt. \qquad (27)$$

Since $a_E(t) = b(t)a(t)$, Eq. (26) allows to obtain the scale factor in the Einstein frame in terms of t and, therefore, in terms of τ trough Eq. (27). In such a way, taking into account Eqs. (18) and (27), one can known, in principle, the explicit form of $\tilde{\varphi}(\tau)$. Unfortunately, it is not possible to obtain an analytic solution for $\tau(t)$,

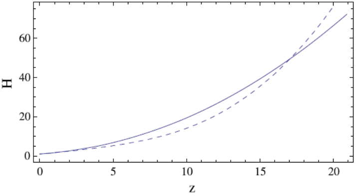

Fig. 1. Comparison of the Hubble parameter, $H(z)$ in the Jordan frame and in the Einstein frame (dashed line), where the Hubble parameter in the Einstein frame has been normalized with its current value.

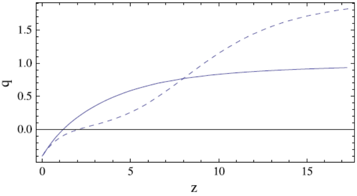

Fig. 2. Comparison of the deceleration parameter, $q(z)$ in the Jordan frame and in the Einstein frame (dashed line).

(27), one can known, in principle, the explicit form of $\varphi(\tau)$. Unfortunately, it is not possible to obtain an analytic solution for $\tau(t)$, but we can perform a complete analytic study in terms of t, noting that, in the Einstein frame, it is only an arbitrary parameter and not the cosmic time. We thus maintain the dot for derivation with respect to t and write explicitly the derivatives w.r.t. the cosmic time τ. This procedure will not affect the final results, because they will be set in terms of the redshift, which is an observable quantity.

Taking into account that $a_E(t) = b(t)a(t)$, we get the Hubble parameter in the Einstein frame

$$H_E(t) = \frac{\partial_\tau a_E}{a_E} = \frac{1}{b(t)} \frac{\dot{a}_E}{a_E}, \tag{28}$$

and a deceleration factor

$$q_E(t) = -\frac{(\partial_\tau^2 a_E) a_E}{(\partial_\tau a_E)^2} = -\frac{\ddot{a}_E a_E}{\dot{a}_E^2} + \frac{\dot{b} a_E}{b \dot{a}_E}. \tag{29}$$

Since the redshift can also be defined in terms of the parameter t,

$$z_E(t) = -1 + \frac{a_{E,0}}{a_E(t)}, \tag{30}$$

where $a_{E,0}$ is the current scale factor, we can eliminate the (unphysical) parameter t, by considering couples of parametric equations. In order to perform this study, we must fit $t_0 = t(\tau_0)$, and we do that demanding that the dimensionless parameter $q_{E,0} = -0.4$ as it was required in the Jordan frame, setting the value $t_0 \simeq 1.24$. Figs. 1 and 2 show that the Hubble parameter $H(z)$ and the deceleration parameter $q(z)$, respectively, are different in the Jordan and Einstein frames. This means that the frames are not physically equivalent (in fact, it would be enough that one of these physical functions were different in the two frames).

One can also compare the dimensionless quantity $\Omega_{m,0}$ in both frames. In the Jordan frame, one can easily see, from the

Fig. 2. Comparison of the deceleration parameter, $q(z)$ in the Jordan frame and in the Einstein frame (dashed line).

00-component of Eq. (2), that it must be defined as $\Omega_{m,0} = \rho_{m,0}/(6f'(R)H_0^2)$ and takes a value compatible with the baryonic component of the Universe, i.e., around 0.04. This parameter is defined in the Einstein frame as $\tilde{\Omega}_{m,0} = \tilde{\rho}_{m,0}/(3H_{E,0}^2)$, and takes a value which is more than twice the value in the Einstein frame, that is $\tilde{\Omega}_{m,0} \simeq 0.09$. On the other hand, in the Einstein frame there is an interaction term which produces $\tilde{\Omega}_{int,0} = (1/b - 1)\tilde{\rho}_{m,0}/(3H_{E,0}^2) = -0.0567$, therefore its absolute value is more than one half the value of the matter component, so it should produce some observable effect.

In order to show even more clearly than the Jordan and Einstein frames are not equivalent, we illustrate this fact in the following way. Let us consider two different researchers studying the model presented in Section 2 following two different routes. One of them refers all its calculations to the original Jordan frame and conclude that this model can describe the distance modulus data, as it is shown in [30]. The other one considers that the Jordan frame and the Einstein frame are physically equivalent and calculate also the distance modulus, but in the Einstein frame. As it is shown in Fig. 3, they obtain different functions. Since the function calculated in the Jordan frame fits the mentioned data, while the function obtained in the Einstein frame does not, the second research would conclude that the model does not describe our Universe, whereas the first one would continue with his study.

5. Conclusions

In this Letter, we have shown that the Jordan and Einstein frames could not be physically equivalent according to the choice of observable quantities. We have consider a particular $f(R)$-model and the resulting model in the Einstein frame, obtained by a

My Study Work

1. I check the solution a(t) in this paper by using Mathematica, but it seems not the sol. of the eq. of motion which authors gave in this paper.
2. I cannot solve thess non-linear ODEs(system) till now.
3. Moreover, I sketch the H-z and q-z diagram which are using the solution of this paper, but finally, they are different. (Althouh in my diagrams, the curves of JF and EF also different.)

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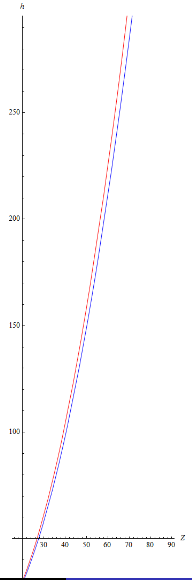

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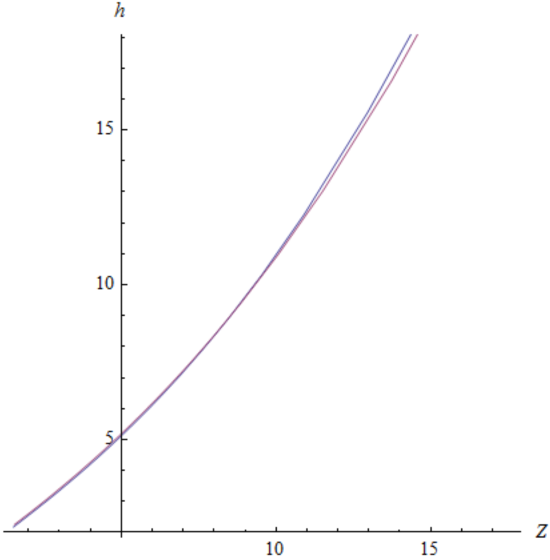

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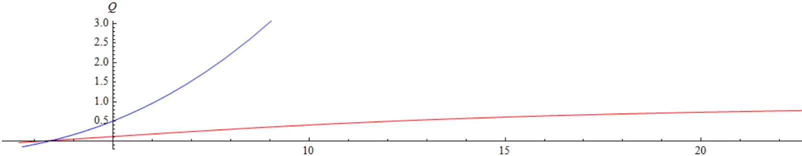

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Outlook