CH. 12 The Renomalization Group. 1

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12.1. Wilson's Approach to Renormalization Theory.

Goal: Illustrate the origin of ultraviolet divergence by isolating the dependence of the functional integral on the short-distance degree of freedom of the field.

1) In Sec 9.2 we obtained the Green function of ϕ^4 theory in terms of functional integral representation of generating functional $Z[J]$. The integration variable are the Fourier components of the field $\phi(k)$.

$$\Rightarrow Z[J] = \int D\phi \, e^{i\int[\mathcal{L}+J\phi]}$$

$$= \left(\prod_k \int d\phi(k)\right) e^{i\int[\mathcal{L}+J\phi]} \qquad (12.1)$$

In order to impose a sharp UV cutoff Λ, we restrict the number of the integration variables displayed in (12.1). i.e. Integrate over $\phi(k)$ with $|k| \le \Lambda$, and set $\phi(k) = 0$ for $|k| > \Lambda$

2) Also, in chapter 8, we approximate the correlation function of the spin field by the propagator of a Euclidean ϕ^4 theory.

$$\longrightarrow \langle s(x)\, s(0)\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2+m^2} \underset{|x|\to\infty}{\sim} e^{-m|x|} \qquad (12.2)$$

As temperature is far from T_c, the size of the mass m, is determined by the one natural scale in this problem, the atomic spacing. ∴ Expect $m \simeq \Lambda$. In our calculation, we let $m \ll \Lambda$ (by adjust relavent parameters,

◎ Integrating over a single momentum shell.

Now, we can carry out the integration over the high-momentum d.o.m. of ϕ, and write the functional integ (12.1) more explicity for the case of ϕ^4 theory.

Apply the cutoff prescription described earlier, and set $J=0$ for simplicity.

bare parameters
∴ no counterterms.

$$\Rightarrow Z = \int [D\phi]_\Lambda \exp\left(-\int d^dx \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right]\right)$$

$$\text{where } [D\phi]_\Lambda = \prod_{|k|<\Lambda} d\phi(k) \qquad (12.4)$$

Now, divide the integration variables $\phi(k)$ into two group

Choose a fraction $b<1$.

The variables $\phi(k)$ with $b\Lambda \leq |k| < \Lambda$ are the high-momen degree of freedom that can be integrate over.

To label these d.o.f., let's define

$$\hat{\phi}(k) = \begin{cases} \phi(k) & \text{for } b\Lambda \leq |k| < \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

Define a new field $\phi(k)$, which is identical to the old for $|k| < b\Lambda$ and zero for $|k| > b\Lambda$.

Then we can replace the old ϕ in the Lagrangian with $\phi + \hat{\phi}$, and rewrite (12.3)

$$\Rightarrow Z = \int D\phi \int D\hat{\phi} \exp\left(-\int d^dx \left[\frac{1}{2}(\partial_\mu\phi + \partial_\mu\hat{\phi})^2 + \frac{1}{2}m^2(\phi+\hat{\phi})^2 + \frac{\lambda}{4!}(\phi+\hat{\phi})^4\right]\right)$$

$$= \int D\phi \left(e^{-\int \mathcal{L}(\phi)}\right) \int D\hat{\phi} \exp\left(-\int d^dx \left[\frac{1}{2}(\partial_\mu\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2 + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\right]\right) \qquad (12.5)$$

* I've gathered all terms indep. of $\hat{\phi}$ into $\mathcal{L}(\phi)$ and the quadratic terms of the form $\phi\hat{\phi}$ vani[sh] since Fourier components of different wavelengt[hs] are orthogonal.

3) <u>Goal</u>: Explain how to perform the integral ove[r]

Transform (12.5) into $Z = \int [D\phi]_{b\Lambda} \exp\left(-\int d^d x \, \underbrace{\mathcal{L}_{eff}}\right)$

only dep. on [ϕ] with $|k| < b\Lambda$ $= \mathcal{L}(\phi) +$ corre[ctions]

∵ We interest in $m^2 \ll \Lambda^2$, and treat the mass term $\frac{1}{2} m^2 \hat{\phi}^2$ as a perturbation.

∴ The leading-order term in the portion of the Lagrangian involving $\hat{\phi}$ is

$$\int \mathcal{L}_0 = \frac{1}{2} \int_{b\Lambda \le |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \hat{\phi}^*(k) \, k^2 \hat{\phi}(k) \qquad (12.7)$$

$$\Rightarrow \overbrace{\hat{\phi}(k)\hat{\phi}(p)} = \frac{\int D\hat{\phi} \, e^{-\int \mathcal{L}_0} \hat{\phi}(k)\hat{\phi}(p)}{\int D\hat{\phi} \, e^{-\int \mathcal{L}_0}}$$

$$= \frac{1}{k^2} (2\pi)^d \delta^{(d)}(k+p) \, \Theta(k) \qquad (12.8)$$

← Various contributions from the perturbati[on] can be ev[aluated] by using W[ick's theorem]

$$\text{where } \Theta(k) = \begin{cases} 1 & \text{if } b\Lambda \le |k| < \Lambda \\ 0 & \text{otherwise} \end{cases} \qquad (12.9)$$

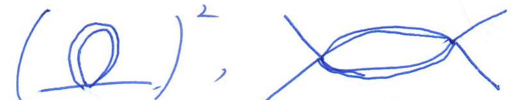
* Consider $\phi^2 \hat{\phi}^2$ term in (12.5)

$$\Rightarrow -\int d^d x \frac{\lambda}{4} \phi^2 \overbrace{\hat{\phi}\hat{\phi}} = \frac{-1}{2} \int \frac{d^d k_1}{(2\pi)^d} \mu \, \phi(k_1) \phi(-k_1) \qquad (12.10)$$

$$\mu = \frac{\lambda}{2} \int_{b\Lambda \le |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}$$

$$= \frac{\lambda}{(4\pi)^{d/2} \Gamma(\frac{d}{2})} \frac{1 - b^{d-2}}{d-2} \Lambda^{d-2} \qquad (12\ldots)$$

(12.10) could just as well have arisen from an expansion of the exponential $\exp\left(-\int d^dx\,\frac{1}{2}\mu\phi^2+\cdots\right)$. (12.12)

e.g. of (12.10) $\Rightarrow$  (12.13)

disconnected diagram $\downarrow$ λ^2

correction term for ϕ^4

Consider the limit in which momenta carried by ϕ are very small compared to $b\Lambda$. Then this diagram has the value $\frac{-1}{4!}\int d^dx\,\zeta\phi^4$ (12.14)

$$= -4!\,\frac{2}{2!}\left(\frac{\lambda}{4}\right)^2\int_{b\Lambda\le|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2$$

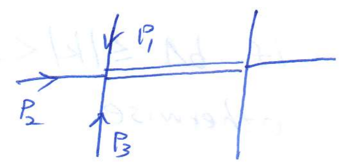
Taylor expand for "∂_μ" interaction terms

$$\frac{-1}{4}\int d^dx\,\eta\,\phi^2(\partial_\mu\phi)^2 \quad (12.17)$$

$$= \frac{-3\lambda^2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\frac{(1-b^{d-4})}{d-4}\Lambda^{d-4}$$

$$\longrightarrow \frac{-3\lambda^2}{16\pi^2}\log\left(\frac{1}{b}\right) \quad (12.15)$$

* We've not only generates contributions $\propto \phi^2, \phi^4$ but higher order of ϕ.

e.g. ϕ^6: 

$\propto \frac{\lambda^2\,(i)\,(p_1+p_2+p_3)}{(p_1+p_2+p_3)^2}$ (12.16)

* By the same combinatoric argument in (4.52), we can rewrite the sum of the series as the exponential of the sum of the connected diagrams. This leads to (12.6) with

$$\mathcal{L}_{eff} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + \left(\text{sum of connected diagrams}\right)$$

Renormalization Group Flow

1) Rescale distances and momenta in (12.6) according t

$$k' = k/b \ , \quad x' = xb \qquad (12.9)$$

integrated over $|k|<\Lambda$

express the explicit form of (12.18).

$$\Rightarrow \int d^dx \, \mathcal{L}_{eff} = \int d^dx \left[\frac{1}{2}(1+\Delta Z)(\partial_\mu\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C(\partial_\mu\phi)^4 + \Delta D\phi^6 + \cdots \right] \quad (12.2$$

In terms of rescaled variable x',

$$\Rightarrow \int d^dx \, \mathcal{L}_{eff} = \int d^dx' \, b^{-d} \left[\frac{1}{2}(1+\Delta Z)b^2(\partial'_\mu\phi)^2 + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C b^4(\partial'_\mu\phi)^4 + \Delta D\phi^6 + \cdots \right] \quad (12.2$$

(12.21) will give rise to the same propagator$_{(12.8)}$

if we rescale the ϕ according to $\phi' = \left[b^{2-d}(1+\Delta Z)\right]^{\frac{1}{2}}\phi$. (12.22)

*. After this rescaling the various perturbations undergo a transformation:

$$\int d^dx \, \mathcal{L}_{eff} = \int d^dx' \left[\frac{1}{2}(\partial'_\mu\phi')^2 + \frac{1}{2}m'\phi'^2 + \frac{1}{4!}\lambda'\phi'^4 + C'(\partial'_\mu\phi')^4 + D'\phi'^6 + \cdots \right] \quad (12.23)$$

The new parameter of the Lagrangian:

iteration equations

$$\begin{cases} m'^2 = (m^2+\Delta m^2)(1+\Delta Z)^{-1}b^{-2} \\ \lambda' = (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}b^{d-4} \\ C' = (C+\Delta C)(1+\Delta Z)^{-2}b^{d} \\ D' = (D+\Delta D)(1+\Delta Z)^{-3}b^{2d-6} \\ \vdots \end{cases} \quad (12.24)$$

and so on and so forth~

The origional $\mathcal{L}$ had $C=D$ but the same eq. will appl if $C, D \neq 0$.

2) The simplest case to consider is a

in the vicinity of the point $m^2 = \lambda = C = D = \cdots = 0$, where all the perturbations vanish. This pint is left unchanged under renormalization group transformation: the free-field Lagrangian $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2$ (12.27) is a fixed point of the transform.

In the vicinity of $\mathcal{L}_0$, we can ignore the terms Δm^2, $\Delta\lambda$ and so on. This give a specially simple transformation law :

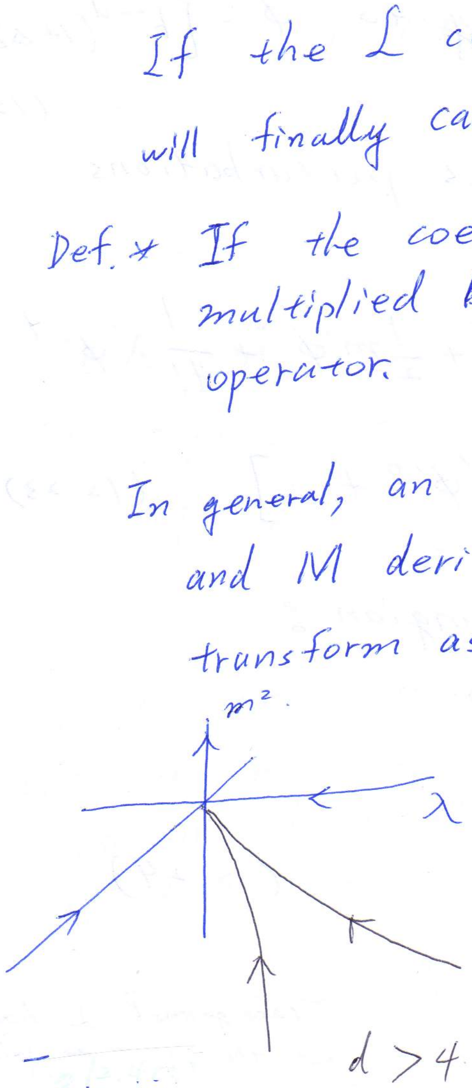
$$m'^2 = m^2 b^{-2}, \quad \lambda' = \lambda b^{d-4}, \quad C' = C b^d, \quad D' = D b^{\cdots}$$

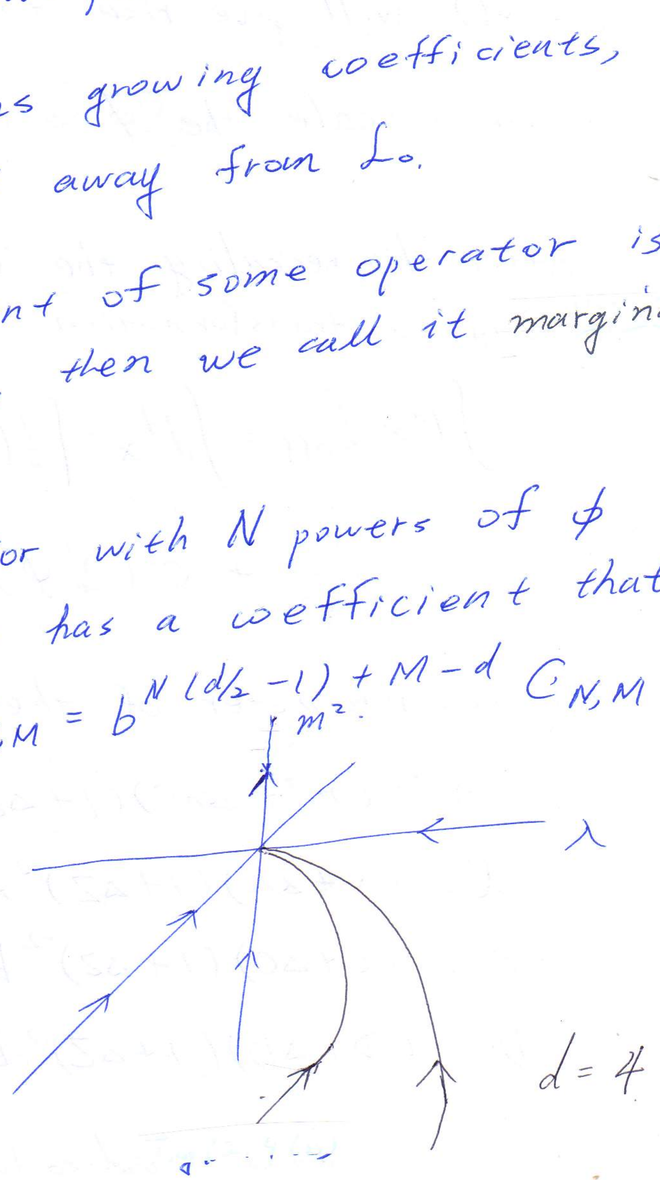
Since $b < 1$, these parameters that are multiplied by negative powers of b grow, and those that are multiplied by positive powers of b decay.

If the $\mathcal{L}$ contains growing coefficients, the will finally carry it away from $\mathcal{L}_0$.

Def. * If the coefficient of some operator is multiplied by b^0, then we call it marginal operator.

In general, an operator with N powers of ϕ and M derivatives has a coefficient that transform as $C'_{N,M} = b^{N(d/2-1)+M-d} C_{N,M}$





3) Consider $d=4$:

In this case (12.26) doesn't give enough information

wherther ϕ^4 is important or not.

$\therefore$ We must bact to the complete transformation law (12.24). The leading contribution to ΔZ is of order λ^2 and can be neglected. $\therefore$

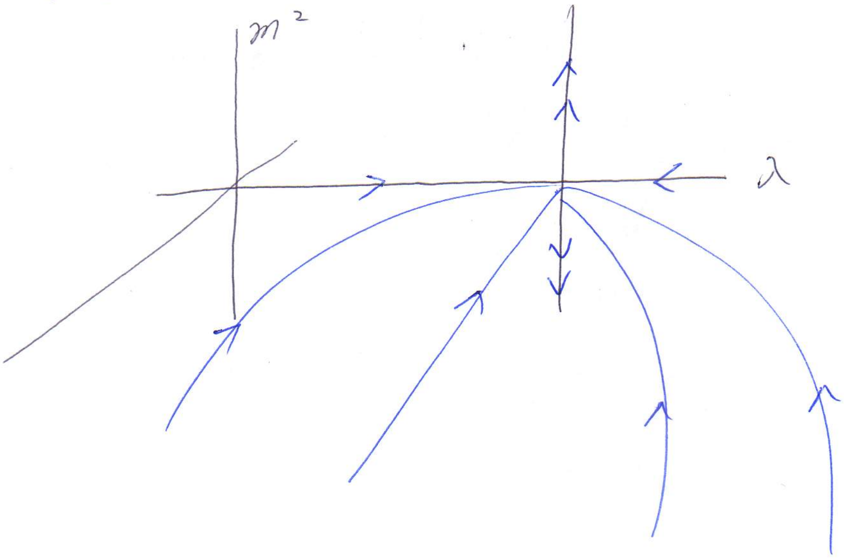
$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \log\left(\frac{1}{b}\right) \qquad (12.28)$$

i.e. λ slow decreases as integra out high-momentum d.o.f.

$\downarrow$

This picture has the puzzling that 4D interating ϕ^4 theory doesn't exist in the limit in which the cutoff goes to infinity.

4) Consider $d<4$:



Renormalization group flows near the free-field fi point in scalar fie theory.

If we include the effect at large distances in d
we derive the recursion formula

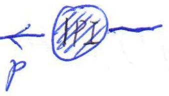
$$\lambda' = \left[\lambda - \frac{3\lambda^2}{(4\pi)^{d/2}\Gamma(\frac{d}{2})} \frac{b^{d-4}-1}{4-d} \Lambda^{d-4}\right] b^{d-4} \qquad (12.29$$

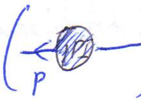
12.2. The Callan - Symanzik Equation. (The C.S.-eq.)

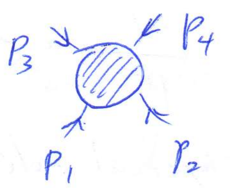
1) Renormalization Conditions

To avoid the singularities (e.g. $m \to 0$), we choose an arbitrary momentum scale M and impose the renormalization conditions at a spacelike momentum p with $p^2 = -M^2$

M ← renormalization scale.


$= 0$ at $p^2 = -M^2$

$\frac{d}{dp^2}\Big($  $\Big) = 0$ at $p^2 = -M^2$ (12-30)


$= -i\lambda$ at $(p_1+p_2)^2 = (p_1+p_3)^2 = (p_1+p_4)^2 = -M^2$

→ implies the two-point func. has a coefficient of 1 at the unphysical momentum $p^2 = -M^2$ rather than on shell ($p^2=0$):

$$\langle\Omega|\phi(p)\phi(-p)|\Omega\rangle = \frac{i}{p^2} \text{ at } p^2 = -M$$

↑ renormalized field

↓ related to bare field ϕ_0: $\phi = \frac{\phi_0}{\sqrt{Z}}$

and $\langle\Omega|\phi_0(p)\phi_0(-p)|\Omega\rangle = \frac{iZ}{p^2}$ at $p^2 =$

with the same relation in Ch. 10

$$\Rightarrow \delta_Z = Z - 1.$$

(but, δ_Z and δ_λ must be adjusted to maintain the new conditions (12.30))

2) Consider the One-loop correction in Yukawa theory. In sec 10.2

we found an expression of the form

 $\sim \dfrac{\Gamma(1-d/2)}{\Delta^{1-d/2}}$ (12.32)

Δ is a linear combination of the fermion mass m_f and p^2.

has a pole at $d=2$ (indep. of p^2); but can be canc by

$\downarrow$

$\therefore$ we can analytical continuation to d

$\downarrow$

$$\sim -p^2\left(\frac{1}{2-d/2} + \log\frac{-1}{p^2} + C\right) \quad (12.3$$

To emphasize the physical role of th cutoff, we will write expressions the form:

$$-p^2\left(\log\frac{\Lambda^2}{-p^2} + C\right) \quad (12.34)$$

⊙ The Callan-Symanzik equation

1) The renormalized Green's functions are numerically equal to the bare Green's functions, up to a rescali by powers of the field strength renormalization Z:

$$\langle\Omega|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle = Z^{-n/2}\langle\Omega|T\phi_0(x_1)\phi_0(x_2)\cdots$$ (1

Let $G^{(n)}(x_1,\cdots,x_n)$ be the connected n-point fn. computed in renormalized perturbation theory.

$$G^{(n)}(x_1,\cdots,x_n) = \langle\Omega|T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle_{\text{connected}} \quad (12$$

$\exists$ a corresponding shift in the coupling const. and field st s.t. the bare Green's fn. remained fixed:

① $M \to M+\delta M$ ③ $\phi \to (1+n\,\delta\eta)\,G^{(n)}$ (12.37)

② $\lambda \to \lambda+\delta\lambda$

$\therefore \Rightarrow$ ④ $G^{(n)} \to (1+n\,\delta\eta)\,G^{(n)}$ ..

$$\Rightarrow dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda$$

$$= n\,\delta\eta\, G^{(n)} \qquad (12.38)$$

To define the dimensionless parameters

$$\underset{\text{def.}}{\beta :=} \frac{M}{\delta M}\delta\lambda, \quad \underset{\text{def.}}{\gamma :=} \frac{-M}{\delta M}\delta\eta \qquad (12.39)$$

$\therefore$ obtain $\Big[M\frac{\partial}{\partial M} + \underbrace{\beta\frac{\partial}{\partial\lambda} + n\gamma}_{\text{indep. } x_i}\Big]\underset{\text{is renormalized.}}{G^{(n)}}(x_1, \cdots, x_n; M, \lambda) = 0$ (12.4

Conclude :

$\forall$ Green's func. of massless ϕ^4 theory must

satisfy $\underbrace{\Big[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\Big]G^{(n)}(\{x_i\}; M, }_{\text{The Callan-Symanzik eq.}}$ (12.

in general, for multiple field theory

e.g. $\Big[M\frac{\partial}{\partial M} + \beta(e)\frac{\partial}{\partial e} + n\gamma_2(e) + m\gamma_3(e)\Big]G^{(n,m)}(\{x_i\}; M, e) =$ (12.42)

for QED

n, m : the number of e^- and photon fields in the Green's func., γ_2 and γ_3 are rescal functions of the e^- a photons

⊙ Computation of β and γ

1) To calculate β, we apply the Callan-Symanzik equation to the four-point function:

$$\left[M\frac{\partial}{\partial M}+\beta(\lambda)\frac{\partial}{\partial\lambda}+4\gamma(\lambda)\right]G^{(4)}(p_1,\cdots,p_4)=0 \quad (12.43)$$

in ϕ^4

Borrowing our result (10.21) from sec 10.2. we can write $G^{(4)}$ as

$$G^{(4)}=\left[-i\lambda+(-i\lambda)^2\left[iV(s)+iV(t)+iV(u)\right]-i\delta_\lambda\right]\cdot\prod_{i=1,\cdots,4}\frac{i}{p_i^2}$$

* (12-30) requires that the correction terms cancel at $s=t=u=-M^2$. The order-λ^2 vertex counterterm is

$$\delta_\lambda=(-i\lambda)^2\cdot 3V(-M^2)$$

$$=\frac{3\lambda^2}{2(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})}{(x(1-x)M^2)^{2-d/2}} \quad (12.44)$$

* The last expression follows from setting $m=0$ and $p^2=-M^2$ in (10.23) for $V(p^2)$.

As $d\to 4$. (12.44) becomes

$$\delta_\lambda=\frac{3\lambda^2}{2(4\pi)^2}\left[\frac{1}{2-d/2}-\log M^2+\underbrace{\text{finite}}_{\text{indep. on } M}\right] \quad (12.45)$$

gives $G^{(4)}$ its M dependence:

$$M\frac{\partial G^{(4)}}{\partial M}=\frac{3i\lambda^2}{(4\pi)^2}\prod_i\frac{i}{p_i^2}$$

Then the C.S.-eq. (12.43) can be satisfied to order λ^2 only if the β function of ϕ^4-theory is given by

$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}+O(\lambda^3) \quad (12.46)$$

Consider the C.S. eq. for the two point function:

$$\left[M\frac{\partial}{\partial M}+\beta(\lambda)\frac{\partial}{\partial\lambda}+2\gamma(\lambda)\right]G^{(2)}(p)=0 \qquad (12.47)$$

$\because$ To ~~one-loop~~ order $\nexists$ propagator corrections to $G^{(2)}$, no dep. on M and λ is introduced to order λ.

$\therefore$ The γ function is zero to this order:

$$\gamma = 0 + O(\lambda^2). \qquad (12.48)$$

In any renormalizable massless scalar field theory, the two point Green's function has the generic form.

$$\begin{aligned} G^{(2)}(p) &= \text{———} + (\text{loop diagrams}) + \text{—}\otimes\text{—} + \cdots \\ &= \frac{i}{p^2}+\frac{i}{p^2}\left(A\log\frac{\Lambda^2}{-p^2}+\text{finite}\right) \\ &\quad + \frac{i}{p^2}(ip^2\delta_Z)\frac{i}{p^2}+\cdots \qquad (12.49) \end{aligned}$$

Apply to CS eq. (neglect β term)

$\Rightarrow$

$$\frac{-i}{p^2}M\frac{\partial}{\partial M}\delta_Z + 2\gamma\frac{i}{p^2}=0 \qquad (12.50)$$

or $\gamma = \frac{1}{2}M\frac{\partial}{\partial M}\delta_Z$ (to lowest order)

* Mind that. $\delta_Z = A\log\left(\frac{\Lambda}{M}\right)^2 + \text{finite}$.

γ is the coeffieient of the logrithm: $\gamma = -A$ (to lowest order)

⊙ Taking propagator corrections into account, the full connected Green's function, to one-loop order, has the general form:

↓

$$G^{(n)} = \begin{pmatrix}\text{tree-level}\\ \text{diagram}\end{pmatrix} + \begin{pmatrix}\text{1PI loop}\\ \text{diagrams}\end{pmatrix} + \begin{pmatrix}\text{Vertex}\\ \text{counterterm}\end{pmatrix}.$$

$$+ \begin{pmatrix}\text{external leg}\\ \text{corrections}\end{pmatrix}$$

$$= \left(\prod_i \frac{i}{p_i^2}\right)\left[-ig - iB\log\left(\frac{\Lambda^2}{-p^2}\right) - i\delta_g + (-ig)\sum_i \left(A_i \log\frac{\Lambda^2}{-p_i^2} - \delta_{Z_i}\right)\right]$$

$$+ \text{finite terms}, \quad (12.52)$$

Applying the C.S. eq., we obtain that.

$$M\frac{\partial}{\partial M}\left(\delta_g - g\sum_i \delta_{Z_i}\right) + \beta(g) + g\sum_i \frac{1}{2} M\frac{\partial}{\partial M}\delta_{Z_i} = 0$$

or $\beta(g) = M\frac{\partial}{\partial M}\left(-\delta_g + \frac{1}{2}g\sum_i \delta_{Z_i}\right)$ (to lowest order). (12.53)

* Note that $\delta_g = -B\log\left(\frac{\Lambda}{M}\right)^2 + \text{finite}.$

$\therefore$ The β function is just a combination of the coefficients of the divergent logrithms:

$$\beta(g) = -2B - g\sum_i A_i \quad \text{(to lowest order)} \quad (12.54)$$

A result:

* To compute the leading term of CS-functions, we needn't be too precise in specifying renormalization conditions! Any momentum scale of order M^2 will yield the same result.

e.g. in Yukawa theory, the tree-level expression:

for the 3-point func is

$$\frac{i}{\not p_1}\frac{i}{\not p_2}\frac{1}{p_3^2}(-ig) \quad (12.55)$$

e.g. 2. in QED., (7.74). the general form of the photon propagator in Feynman gauge is

$$D^{\mu\nu}(q) = \underline{\underline{D(q)}}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \frac{-i}{q^2}\frac{q^\mu q^\nu}{q^2} \quad (12.56)$$

Satisfy CS- eq.

$\because$ The corrections to this function have the form (12.49), the arguments following the formula. are valid. for photons as well as for electrons and. scalars. $\therefore$ To leading order,

$$\gamma_2 = \frac{1}{2} M\frac{\partial}{\partial M}\delta_2, \quad \gamma_3 = \frac{1}{2} M\frac{\partial}{\partial M}\delta_3 \quad (12.57)$$

Similarly, consider the 3-point connected Green's function, $\langle \bar{\psi}(P_1)\psi(P_2)A_\mu(q)\rangle$, projected onto transverse components of the photon. At the leading order, this function equals

$$\frac{i}{\not{P_1}}(-ie\gamma^\mu)\frac{i}{\not{P_2}}\frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)$$

$\therefore$ (12.53) gives the lowest-order expreesion for β function : $\beta(e) = M\frac{\partial}{\partial M}(-e\delta_1 + e\delta_2 + \frac{e}{2}\delta_3)$ (12.58)

<u>Goal</u> : To find the explicit expression for QED $\because$ we must write down $\delta_1, \delta_2, \delta_3$

Reading from (10.43) and (10.44)

$$\Rightarrow \delta_1 = \delta_2 = \frac{-e^2}{(4\pi)^2}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} + \text{finite}.$$

$$\delta_3 = \frac{-e^2}{(4\pi)^2}\frac{4}{3}\frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}} + \text{finite} \quad (12.59)$$

Using (12.57) and (12.59) we derive the leadi[ng] order $\gamma_2(e) = e^2/16\pi^2$, $\gamma_3(e) = e^2/12\pi^2$.

And from (12.58), we obtain

$$\beta(e) = \frac{e^2}{12\pi^2}. \qquad (12.61)$$

◎ The meaning of β and γ

1) Recall the bare and renormalized field are related by

$$\phi(p) = Z(M)^{-1/2} \phi_0(p) \qquad (12.62)$$

The renormalized field is shifted by δ_η. $\rightarrow$ $\delta_\eta = \frac{Z(M+\delta M)^{-1/2}}{Z(M)^{-1/2}} - 1.$

∴ Our original definition (12.39) of γ gives us immediately

$$\gamma(\lambda) = \frac{1}{2}\frac{M}{Z}\frac{\partial Z}{\partial M} \qquad (12.63)$$

∵ $\delta_Z = Z - 1$ (eq. 10.17).

Similarly, we can find an instructive expression for β in terms of the parameters of bare perturbation theory. ∵ The bare Green's functions depends on the bare coupling λ_0 and the cut off, the definition of β can be rewritten as

$$\beta(\lambda) = M\frac{\partial}{\partial M}\lambda \Big|_{\lambda_0, \Lambda} \qquad (12.64)$$

.3 Evolution of Coupling Constants

◎ Solution of the Callan-Symanzik Equation.

Goal: To solve two-point function, $G^{(2)}(p)$, in scalar field theory.

∵ $G^{(2)}(p)$ has dimensions of $(\text{mass})^{-2}$,

$$\Rightarrow G^{(2)}(p) = \frac{i}{p^2}\, g(-p^2/M^2) \qquad (12.65)$$

Tool: * We can use the variable p to represent the magnitude of the spacelike momentum:

$$p = (-p^2)^{1/2}.$$

Then, we can rewrite the CS eq. as follows:

$$\left[p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda} + 2 - 2\gamma(\lambda)\right] G^{(2)}(p) = 0 \qquad (12.66)$$

In free field theory, β and γ vanish and we recover the trivial result.

$$G^{(2)}(p) = \frac{i}{p^2} \qquad (12.67)$$

* Analogue:

Imagine a pipe with bacteria and with initial density $D_i(x)$. The growth rate and flow rate are functions of x. The problem is to determine the density $D(t,x)$ at all subsequent times.

$$\left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x} - \rho(x)\right] D(t,x) = 0 \qquad (12.68)$$

↳ identical to (12.66) with.

① $\log(p/M) \leftrightarrow t$. ③ $-\beta(\lambda) \leftrightarrow v(x)$ ⑤ $G^{(2)}(p,\lambda) \leftrightarrow D(t,x)$

② $\lambda \leftrightarrow x$ ④ $2\gamma(\lambda) - 2 \leftrightarrow \rho(x)$

Suppose the initial concentration: $D(t,x) = (D_i(x)$ at $t=$

We can find out where it was at time zero by integrating its motion backward in time.

The position of this element at time $t=0$ is given by $\bar{x}(t,x)$, which satisfies the differential eq. $\frac{d}{dt'}\bar{x}(t';x) = -v(\bar{x})$

with $\bar{x}(0;x) = x$ (12.70)

Then, immediately,

$$D(t,x) = D_i(\bar{x}(t,x)) \exp\left(\int_0^t dt' \rho(\bar{x}(t';x))\right) \quad (12.71)$$

$$= D_i(\bar{x}(t;x)) \cdot \exp\left(\int_{\bar{x}(t)}^{x_i} dx' \frac{\rho(x')}{v(x')}\right)$$

$t=0 \Rightarrow -p^2 = M^2$, and initial concentration $D_i(x)$ becomes an unknown function $\hat{g}(\lambda)$.

Then $$\boxed{G^{(2)}(p,\lambda) = \hat{g}(\bar{\lambda}(p;\lambda)) \exp\left(-\int_{p'=M}^{p'=p} d\log(p'/M) \cdot 2[1-\gamma(\bar{\lambda}(p';\ldots$$ (12.72)

↳ We can check (12.72) solve CS-eq. by using identity

$$\int_{\lambda}^{\bar{\lambda}} \frac{d\lambda'}{\beta(\lambda')} = \int_{p'=M}^{p'=p} d\log\left(\frac{p'}{M}\right)$$

where $\bar{\lambda}(p;\lambda)$ solves.

often called renormalization group equation.

$$\frac{d\bar{\lambda}(p;\lambda)}{d\log(p/M)} = \beta(\bar{\lambda}), \quad \bar{\lambda}(M;\lambda) = \lambda \quad (12.73)$$

↳ describe the flow.

↓ as func. of momentu

* The rate of this flow is just β-function

⊙ $\bar{\lambda}(p)$ as running coupling constant. $\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right)\bar{\lambda} = 0$

A simplified form:

$$G^{(2)}(p,\lambda) = \frac{i}{p^2} g(\bar{\lambda}(p;\lambda)) \exp\left(2\int_M^p d\log\left(\frac{p'}{M}\right) \gamma(\bar{\lambda}(p';\lambda))\right) \quad (12.76),$$

To leading order in perturbation theory, for 4-pt. func.,

we have $G^{(4)}(P) \underset{\substack{\text{e.g. } p_i^2 = -P^2 \\ p_i \cdot p_j = 0 \\ s,\,t,\,u \text{ are of order } -P^2}}{=} \left(\frac{i}{P^2}\right)(-i\lambda)$ (12.77)

Using the fact that $G^{(4)}$ has dimensions of $(\text{mass})^{-8}$, we can exchange M for P in the CS-eq. and write this eq. as

$$\left[P\frac{\partial}{\partial P} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 8 - 4\gamma(\lambda)\right] G^{(4)}(P;\lambda) = 0 \quad (12.78)$$

The solution to this equation is

$$G^{(4)}(P;\lambda) = \frac{1}{P^8}\, g^{(4)}(\bar{\lambda}(P;\lambda)) \exp\left(4\int_M^P d\log\left(\frac{P'}{M}\right)\gamma(\bar{\lambda}(P';\lambda))\right) \quad (12.79)$$

This formula must agree with (12.77) to leading order in λ; this matching requires that

$$g^{(4)}(\bar{\lambda}(P;\lambda)) = -i\bar{\lambda} + O(\bar{\lambda}^2) \quad (12.80)$$

As a check on these formal arguments, we can use explicit form of the β function of ϕ^4 theory found in (12.46) and the renormalization group eq. (12.73) to evaluate the running coupling constant of ϕ^4 theory. This running coupling constant satisfies the differential eq. $\frac{d\bar{\lambda}}{d\log(P/M)} = \frac{3\bar{\lambda}^2}{16\pi^2}$ with $\bar{\lambda}(M;\lambda) =$ (12.8

Integrating $\longrightarrow$ $$\frac{\frac{1}{\lambda}-\frac{1}{\bar{\lambda}}}{3/16\pi^2} = \log\left(\frac{P}{M}\right)$$

$$\therefore\ \bar{\lambda}(p) = \frac{\lambda}{1-\left(\frac{3\lambda}{16\pi^2}\right)\log\left(\frac{P}{M}\right)} \qquad (12.82)$$

If we expand the running coupling const. $\bar{\lambda}(p)$ in powers of λ, we find that the successive powers of the coupling constant are multiplied by powers of logrithms.

$$\lambda^{n+1}\left(\log\frac{P}{M}\right)^n \text{ which become large.}$$

and invalidate a simple perturbation expansion for p much greater or much less than M.

⑥ An Application to QED.

$\therefore$ We define the massless limit of QED by a renormalization scale M at which the renormalized coupling e_r is defined.

If $M \longrightarrow m$ (electron)

then $e_r \longrightarrow e$ (physical electron charge)

thus, CS-eq. for the Fourier transform of the potential has no γ term

$$\Rightarrow \left[M\frac{\partial}{\partial M} + \beta(e_r)\frac{\partial}{\partial e_r}\right] V(q; M, e_r) = 0 \qquad (12.83)$$

dimension : $(\text{mass})^{-2}$

$\therefore$ We can trade dependence on M for dependence on q as in the scalar field theory. This gives

$$\left[q\frac{\partial}{\partial q} - \beta(e_r)\frac{\partial}{\partial e_r} + 2 \right] \tilde{V}(q; M, e_r) = 0 \qquad (12.84)$$

(12.84) is almost the same as (12.66)

$\therefore$ We can immediately write down the solution as a special case of (12.76)

$$\tilde{V}(q, e_r) = \frac{1}{q^2}\, \nu\left(\bar{e}(q; e_r)\right) \qquad (12.85)$$

where $\bar{e}(q)$ is the solution of the renormalization group eq.

$$\frac{d\,\bar{e}(q; e_r)}{d\log\left(\frac{q}{M}\right)} = \beta(\bar{e}), \quad \bar{e}(M; e_r) = e_r \qquad (12.86)$$

Comparing this formula for $\tilde{V}(q)$ to the leading-order result $\tilde{V}(q) \simeq \frac{e^2}{q^2}$ we can identify $\nu(\bar{e}) = \bar{e}^2 + O(\bar{e}^4)$.

Then $\tilde{V}(q, e_r) = \bar{e}^2(q; e_r)/q^2$ (12.87)

up to corrections that are suppressed by powers of e_r^2 and contain no compensatory large logrithms of q/M.

Using QED β function (12.61), we can integrate (12.86) to find $\frac{12\pi^2}{2}\left(\frac{1}{e_r^2} - \frac{1}{\bar{e}^2}\right) = \log\left(\frac{q}{M}\right)$

Simplified.

$$\bar{e}^2(q) = e_r^2 \Big/ \left(1 - (e_r^2/6\pi^2)\log(q/M)\right) \qquad (12.8$$

Set $M^2 = Am^2$ and by using $\alpha = e^2/4\pi \Rightarrow \bar{\alpha}(q) = \alpha \Big/ \left[1 - (\alpha/3\pi)\log\left(\frac{-q^2}{Am^2}\right)\right]$ (1

◎ Alternatives for the running coupling constants.

By the arguments of previous pages, the Green's func. in any such theory obey a CS eq. The solution of this eq. dep on a running coupling const., $\bar{\lambda}(p)$, which satisfies a differential eq.

$$\frac{\partial \bar{\lambda}}{\partial \log (P/M)} = \beta(\bar{\lambda}) \qquad (12.90)$$

in which the function $\beta(\lambda)$ is computable as a power series in the coupling const.

* As a matter of principle, three behaviors are possible in the region of small λ:

(i) $\beta(\lambda) > 0$

(ii) $\beta(\lambda) = 0$

(iii) $\beta(\lambda) < 0$

We can classify those theories by examine their coupling const.

Class 1. The running coupling const. becomes large in the region of high momenta and goes to zero in IR, leading to small-momentum behavi

Class 2. The coupling const. doesn't flow.

Class 3. The running coupling const. becomes large in the large-distance regime and becom small at large momenta or short distance

e.g. The sign of the QED β func. were reversed: $\beta(e) = -\frac{1}{2} C e^3$ (12.91)

Following our earlier analysis, we would h

$$\bar{e}^2(p) = \frac{e^2}{1 + Ce^2 \log(P/M)} \qquad (12.92)$$

→ coupling const
→ zero at
log rate as
⇒ So-called asymtotic free

⊙ Note that all of our explicit solutions for running coupling constants – ((12.82), (12.88), (12.92)) – predict that the running coupling constant becomes infinite at a finite value of the momentum p. e.g. According to (12.82), the running coupling const. of ϕ^4 theory should diverge at.

$$p \sim M \exp\left(\frac{16\pi^2}{3\lambda}\right) \quad (12.93).$$

《Discussion》

Case. (i). ∃ a nontrivial fixed point in ϕ^4 theory in $d<4$, and many more examples are known.

For β func. of the form of Fig 12.4 (a), the β func. behaves in the vicinity of the fixed point as

$$\beta \simeq -\underset{>0}{\underset{\uparrow}{B}}(\lambda - \lambda_*) \quad (12.94)$$

For $\bar\lambda$ near λ_*,

$$\underbrace{\frac{d\bar\lambda}{d\log p} \simeq -B(\bar\lambda - \lambda_*)}_{\text{sol.}} \quad (12.95)$$

$$\underbrace{\bar\lambda(p)}_{\left[\text{tends to } \lambda_* \text{ as } p\to\infty\right]} = \lambda_* + C\left(\frac{M}{p}\right)^B \quad (12.96)$$

This behavior has a consequence for the exact solution (12.72) of the CS-eq. for $G(p)$.

For p sufficiently large, the integral in the exp. fac[tor] in this eq. will be determined by values of p for which $\bar\lambda(p)$ is close to λ_*. Then

Anomalo[us] dimensio[n]

$$G(p) \simeq g(\lambda_*) \exp\left[-\left(\log\left(\frac{p}{M}\right)\right)\cdot 2(1-\gamma(\lambda_*))\right] \simeq C\cdot\left(\frac{1}{p^2}\right)^{1-\gamma(\lambda_*)}$$

Case (ii) If the β-func. has the form in Fig 12.4 (b), then the running coupling const. will tend to a fixed point λ_* as $p \to 0$. The 2-pt. correlation func. of fields $G(p)$ will tend to a power law as in (12-97) for asymptotically small momenta.

The two cases shown in Fig (a) and (b) are called (i) ultraviolet - stable

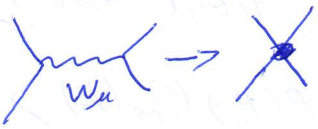
& (ii) infrared- stable fixed points.

4. Renormalization of Local

1) Consider the theory of strongly interacting quarks perturbed by the effect of weak decay processes. Let's write the interaction of the quarks with W_μ very schematically as

$$\delta \mathcal{L} = \frac{g}{\sqrt{2}} W_\mu \bar{\psi} \gamma^\mu (1-\gamma^5)\psi \quad (12.98)$$

$\downarrow$ ~~the~~ propagator.



$$\frac{-i g^{\mu\nu}}{q^2 - m_W^2 + i\epsilon} \quad (12.99).$$

For momentum transfers small compared to m_W, ignore q^2 in the W propagator and write this interaction as the matrix element of the operator

$$\frac{g^2\, \mathcal{O}(x)}{2 m_W^2}, \quad \text{where } \mathcal{O}(x) = \bar{\psi}\gamma^\mu(1-\gamma^5)\psi\bar{\psi}\gamma_\mu(1-\gamma^5)\psi \quad (12.100)$$

2) Q: How to analyze the effects of operator (12.100) on strongly interacting particles composed of quarks and antiquarks?

$\Rightarrow$ To compute the Green's func of the operator $\mathcal{O}$ together with fields that create and destroy quarks. If we compute the theory of quarks by a theory of free fermions, it's easy to compute these Green's functions;

e.g. $\langle \psi(p_1)\bar{\psi}(-p_2)\psi(p_3)\bar{\psi}(-p_4)\mathcal{O}(0)\rangle$

$$= S_F(p_1)\gamma^\mu(1-\gamma^5)S_F(p_2)S_F(p_3)\gamma_\mu(1-\gamma^5)S_F(p_4) \quad (12.101)$$

3) The renormalized operator O_M is a rescaled version of the operator O_0 built of bare fields,

$$O_0(x) = \bar{\psi}_0 \gamma^\mu (1-\gamma^5)\psi_0 \bar{\psi}_0 \gamma_\mu (1-\gamma^5)\psi_0 \qquad (12.102)$$

As we did for the elementary fields, we can write this relation as $O_0 = Z_O(M) O_M$ (12.103)

Lets return to the scalar field and consider $O(x)$ to be a local operator in a scalar field theory.

Define. $G^{(n;1)}(p_1, \cdots, p_n; k) = \langle \phi(p_1) \cdots \phi(p_n) O_M(k) \rangle$ (12.104)

related to a G-func. of bare fields

$$G^{(n;1)}(p_1, \cdots, p_n; k) = Z(M)^{-n/2} Z_O(M)^{-1} \langle \phi_0(p_1) \cdots \phi_0(p_n) O_0 \qquad (12.105)$$

Repeating the derivation of (12.63) and (12.64) we find that the Green's functions containing a local operator obey the CS-eq. $\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda) + \gamma_O(\lambda)\right] G^{(n;1)} = 0$

(12.106)

where $\gamma_O = M\frac{\partial}{\partial M}\left(\log Z_O(M)\right)$ (12.107)

For such a set of operators $\{O^i\}$, the relation of renormalized and bare operators must be generalized to $O_0^i = Z_O^{ij}(M) O_M^j$. (12.108)

The relation in turn implies that the anomalous dimension function γ_O in the CS-eq. must be generalized to a matrix, $\gamma_O^{ij} = \left[Z_O^{-1}(M)\right]^{ik} M\frac{\partial}{\partial M}\left[Z_O(M)\right]^{kj}$

(12.109)

If O is the quark number current $\bar{\psi}\gamma^\mu\psi$,

its normalization is fixed once and for all because the associated charged $Q = \int d^3x\, \bar{\psi}\gamma^0\psi$

is just the conserved integer number of quarks minus antiquarks in a given state,

e.g. In QED, the specific linear combination

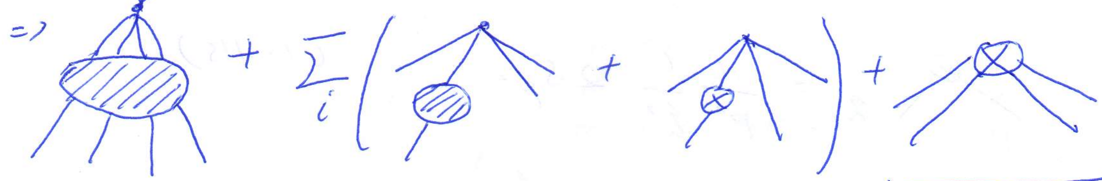
$$T^{\mu\nu} = \frac{1}{2}\bar{\psi}\left[\gamma^\mu D^\nu + \gamma^\nu D^\mu\right]\psi - F^{\mu\lambda}F^\nu{}_\lambda. \quad (12.110),$$

receives no rescaling and no anomalous dimension.

4) To find a simple formula for γ_O, we follow the same path that took us from (12.52) to (12.53) for the β function. Consider an operator whose normalization condition is based on a Green's function with m scalar fields:

$$G^{(m;1)} = \langle \phi(p_1)\cdots\phi(p_m)\, O_M(k)\rangle \quad (12.111)$$

Compute this G-func. to one-loop order,

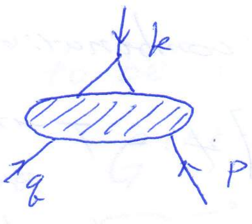


countertorm δ_O needed to maintain the renormalization condition

and assume this sum satisfy CS-eq, to leading order in λ, then

$$\gamma_O(\lambda) = M\frac{\partial}{\partial M}\left(-\delta_O + \frac{m}{2}\delta_Z\right) \quad (12.112).$$

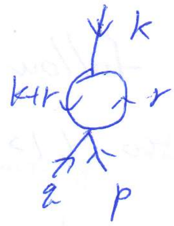
5) To clarify the distinction between the underlying m which is renormalized to zero, and the explicit mass perturbation, we will analyze a Gree's function of ϕ^2 in which this operator carries a specific nonzero momentum. Thus, choose to define the renormalization of ϕ^2 by the convention



$$= \langle \phi(p)\phi(q)\phi^2(k)\rangle = \frac{i}{p^2}\frac{i}{q^2}\cdot 2 \qquad (12.113)$$

at $p^2 = q^2 = k^2 = -M^2$

6) The 1-PI one-loop correction to (12.113):



$$= \frac{i}{p^2}\frac{i}{q^2}\int\frac{d^d r}{(2\pi)^d}(-i\lambda)\frac{i}{r^2}\frac{i}{(k+r)^2} \qquad (12.114)$$

$$= \frac{i}{p^2}\frac{i}{q^2}\left[\frac{-\lambda}{(4\pi)^2}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\right]$$

Δ def: a func. of external momenta;

At $-M^2$, this contribution must be canceled by a counterterm diagram,

$$= \frac{i}{p^2}\frac{i}{q^2}\,2\delta_{\phi^2} \qquad (12.115)$$

$$\therefore \delta_{\phi^2} = \frac{\lambda}{2(4\pi)^2}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \qquad (12.116)$$

$\because \delta_Z$ is finite to order λ, this is the only contribution to (12.112) and we find $\gamma_{\phi^2} = \lambda/16\pi^2$ (12.117)

This can be used together with γ and β func. of pure massle ϕ^4 theory to disscuss the scaling of G-func. include mass oper

5. Evolution of Mass Parameters.

1) If $\mathcal{L}_M$ is the massless Lagrangian renormalized at the scale M, the new Lagrangian will be

$$\mathcal{L}_M - \frac{1}{2} m^2 \phi_M^2 \qquad (12.118)$$

In general, we can use the operator $m^2\left(\frac{\partial}{\partial m^2}\right)$ to count the number of insertions l of ϕ^2. Then the Green's functions of the massive ϕ^4 theory, renormalized according to the mass-indep. scheme, satisfy the eq.

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda) + \gamma_{\phi^2} m^2 \frac{\partial}{\partial m^2}\right] G^{(n)}(\{p_i\}; M, \lambda, m^2) \qquad (12.119)$$

↓ extend to any perturbation of massless ϕ^4 theory. In general,

$$\mathcal{L}(C_i) = \mathcal{L}_M + C_i \mathcal{O}^i_M(x) \qquad (12.120)$$

and the Green's functions of this perturbation theory satisfy: $\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + \sum_i \gamma_i(\lambda)\, C_i \frac{\partial}{\partial C_i}\right] \times G^{(n)}(\{p_i\}; M, \lambda, \{C_i\})$

rewrite

$$\mathcal{L}(\rho_i) = \mathcal{L}_M + \rho_i M^{4-d_i} \mathcal{O}^i_M(x) \qquad (12.121)$$

mass dimension of the operator $\mathcal{O}^i$

its size indicate the importance of the corresponding operator at the scale M.

be modified.

$$\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + n\gamma + \sum_i \left[\gamma_i(\lambda) + d_i - 4\right]\rho_i\frac{\partial}{\partial \rho_i}\right] G^{(n)}(\{p_i\}; M, \lambda\{\rho_i\}) = \qquad (12.123)$$

If def. $\beta_i = (d_i - 4 + \gamma_i)\rho_i$ (12.124)

then $\left[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + \sum_i \beta_i \frac{\partial}{\partial\rho_i} + n\gamma\right] G^{(n)}(\{P_i\}; M, \lambda, \{\rho_i\}) =$ (12.125)

Now, all of the coupling const. ρ_i appear on the same footing as λ.

Sole the eq., we will find that the sol. will dep. on a set of running coupling constants which obey

$$\frac{d\bar{\rho}_i}{d\log(p/M)} = \beta_i(\bar{\rho}_i, \bar{\lambda}). \quad (12.126)$$

As all the dimensionless parameters λ, ρ_i are small, so that we are close to the free scalar field $\mathcal{L}$. In this situation, we can ignore the contribution of γ_i to β_i; then

$$\frac{d\bar{\rho}_i}{d\log(p/M)} = [d_i - 4 + \cdots]\bar{\rho}_i \quad (12.127)$$

$\to$ sol: $\bar{\rho}_i = \rho_i\left(\frac{p}{M}\right)^{d_i - 4}$ (12.128)

2) Move one step closer to see 12.1.

$\Rightarrow$ The expansion of the lagrangian about the free scalar field theory $\mathcal{L}_0$ reads:

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2}\rho_m M^2 \phi_M^2 - \frac{1}{4}\lambda M^{4-d}\phi_M^4 + \cdots \quad (12.129)$$

Also, required to change in the formalism is that of recomputing β and γ in new dimension.

e.g. The computation of γ_{ϕ^2}, (12.114).

For general values of d, the derivative of the counterterm δ_{ϕ^2} with respect to $\log M$ still involves the factor

$$M\frac{\partial}{\partial M}\left(\frac{\Gamma(2-d/2)}{(M^2)^{2-d/2}}\right) = -2 + O(4-d).$$

hold for all of the γ_i and the β-func. is shifted only by the contribution of the mass dimension of λ.

$\therefore$ for $d \xrightarrow[\text{near}]{} 4$.

(4) are the 4D results and omitted corretion terms are of order $\lambda(d-4)$

$$\beta = (d-4)\lambda + \beta^{(4)}(\lambda) + \cdots$$

$$\beta_m = \left[-2 + \gamma_{\phi^2}^{(4)}\right]\rho_m + \cdots \qquad (12.131)$$

$$\beta_i = \left[d_i - d + \gamma_i^{(4)}\right]\rho_i + \cdots$$

Using 4D result (12.46) for β, we find

$$\beta = -(4-d)\lambda + \frac{3\lambda^2}{16\pi^2} \qquad (12.132)$$

* As λ is large, the coupling const. decreases as a result of its own nonlinear effect. These two tendencies come into balance at the zero of the bata function

$$\lambda_* = \frac{16\pi^2}{3}(4-d) \qquad (12.133).$$

◎ Critical Exponents : A first Look.

As an application of the formalism of this section, calculate the renormalization group flow of the coefficient of the mass operator in ϕ^4 theory. This is found by integrating (12.126) using the value of β_m from (12.131) : $\frac{d\bar{\rho}_m}{d\log p} = \left[-2 + \gamma_{\phi^2}(\bar{\lambda})\right]\bar{\rho}_m$ (12.134)

For $\lambda = 0$, this eq. gives the trivial sol.

$$\bar{\rho}_m = \rho_m \left(\frac{M}{p}\right)^2 \quad (12.135).$$

Recall $\rho_m \equiv (m/M)^2$, the characteristic range of correlations, which in statistical mechanics would be called the correlation length ξ, is given by $\xi \sim p_0^{-1}$, where $\bar{\rho}_m(p_0) \sim 1$ (12.136).

If we evaluate this criterion, we find $\xi \sim (M^2/\rho_m)^{-1/2}$

i.e. $\xi \sim m^{-1}$, as we would have expected.

If we set $\bar{\lambda} = \lambda_*$, then (12.134) has the sol.

$$\bar{\rho}_m = \rho_m \left(\frac{M}{p}\right)^{2-\gamma_{\phi^2}(\lambda_*)} \quad (12.137)$$

This gives a nontrivial relation

$$\xi \sim \rho_m^{-\nu} \quad (12.138)$$

where the exponent ν is given ~~formally~~ by the expression $\nu = \dfrac{1}{2 - \gamma_{\phi^2}(\lambda_*)}$ (12.139)

Using (12.133) and (12.117) we can evaluate this for d near 4: $\nu^{-1} = 2 - \frac{1}{3}(4-d)$ (12.140)

* In statistical mechanics, ρ_m is proportional to the deviation from the critical temperature, $(T - T_c)$. Then, the correlation length in a magnet grows as $T \to T_c$ according to the scaling relation $\xi \sim (T - T_c)^{-\nu}$ (12.141)

A scaling behavior of the type (12.141) is observed in magnets, at it's known that several definite scaling law occur, depending on the symmetry of the spin ordering.

In Ch. 11, we considered a generalization of ϕ^4 theory to a theory of N fields with O(N) symmetry. If corresponding with it, then (12.140) gives a prediction for the value of ν in magnets with prefered axis. In sec 13.1, we can repeat the analysis leading to this equation in the O(N)-symmetric ϕ^4 theory and derive the formu

$$\nu^{-1} = 2 - \frac{N+2}{N+8}(4-d) \qquad (12.142)$$

valid for general N to 1st-order in (4-d).

For the case N = 1, 2, 3 and d = 3, this formula predicts $\nu = 0.60, 0.63, 0.65.$ (12.143) The best experimental determinination of ν in magnetic system give

$$\nu = 0.64,\ 0.67,\ 0.71. \text{ for } N = 1, 2, 3. \qquad (12.144)$$

$\therefore$ The prodiction (12.143) gives a reasonable first approximation to the experimental results.