Basics of Fundamental Group

Math 450

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Abstract

The goal of this project is twofold:
Firstly, to introduce the basics of Fundamental group including the definition and the motivation befind the fundamental group, to calculate some examples, and to define an induced homomorphism by using a continuous map.

Secondly, applying the functorial property (Corollary 52.5 in Munkres, and we are using its countrapositive statement) to classify the following 47 topological spaces into 10 fundamental groups:

(1) $\mathbb{R}^n$, $n \in \mathbb{Z}^+$

(2) S^1

(3) ~ (8) $\underbrace{A,\ D,\ O,\ P,\ Q,\ R}_{\text{spaces in these shapes}}$

(9) S^2

(10) $S^1 \times S^1$

(11) $[0,1] \subset \mathbb{R}$ (a closed interval)

(12) $\underbrace{S^1 \times \cdots \times S^1}_{n\text{-many}}$, or n-leafed rose space

(13) ~ (31) spaces in these shapes: $\left\{ \begin{array}{l} X,\ Y,\ Z,\ T,\ S,\ C,\ E,\ F, \\ G,\ H,\ I,\ J,\ K,\ L,\ M, \\ N,\ U,\ V,\ W \end{array} \right.$

(32) $\{0\}$ (one-point space)

(33) D^2 (a disk in $\mathbb{R}^n$)

(34) a cone space

(35) a convex space $\subset \mathbb{R}^n$

(36) a star-like space $\subset \mathbb{R}^n$

(37) $\mathbb{R}^3 \setminus$ (an unknot)

(38) $\mathbb{R}^3 \setminus$ (a left handed trefoil)

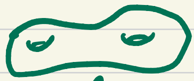
(39) $\mathbb{R}^3 \setminus$ (a right handed trefoil)

(40) a Möbius band

(41) a cylinder

(42) RP^2

(43) a Klein bottle

(44) M_2 - space, T^2; $S^1 \times S^1$ 

$g=2$

(45) M_g space, $g = 1, 2, \dots, n$

(configuration spaces, n-tori)

(46) A figure eight space 

or letter B

(47) $\mathbb{R}^2 \setminus \{0,0\}$

Although if two topological spaces have the same fundamental group then we cannot say they are homeomorphic, if two spaces have different groups, then we can say they are not homeomorphic.

Contents

Part I. Motivations

Goal: Determine whether two topological spaces are homeomorphic.

$S^1 \simeq S^2$?

$\mathbb{R}^1 \simeq \mathbb{R}^2$?

S^1 in $\mathbb{R}^2$ vs S^1 in $\mathbb{R}^2 \setminus \{0,0\}$?

By Funtorial Property

Corollary §2.5 (Munkres)

Or, Thm 12.25 (Starbird-Su)

$X \simeq Y \Rightarrow \pi_1(X) \simeq \pi_1(Y)$

homeomorphic　　isomorphic

$\pi_1(X) \not\simeq \pi_1(Y) \Rightarrow X \not\simeq Y$

But, $\pi_1(X) \simeq \pi_1(Y) \not\Rightarrow X \simeq Y$

Ans: $\pi_1(S^1) = \mathbb{Z} \neq \{e\} = \pi_1(S^2) \Rightarrow S^1 \not\simeq S^2$

$\pi_1(\mathbb{R}^1) = \pi_1(\mathbb{R}^2) = \cdots = \pi_1(\mathbb{R}^n) = \{e\} \Rightarrow$ We cannot use $\pi_1(X)$ to distinguish $\mathbb{R}$ and $\mathbb{R}^2$.

Why studying fundamental group $\pi_1(X)$?

1. It's an eventual application of investigeting spaces like complements of knots.

2. It can give an algebraic topo way (as an alternative) to prove Fundamental Theorem of Algebra without using analysis tools (in complex analysis), i.e. by using fundamental group and winding number.

§56 in Munkres

3. Henri Poincaré defined the fundamental group in 1895 and had his conjecture five years later (1900):
   The only solution to $\pi_1(M^3) = 1$ is $M^3 = S^3$.

4. Thm 12.56 (Starbird-Su)

Each 2-manifold in the following infinite list is topologically different from all the others on the list:

$$S^2, \ \#_{i=1}^{n} \mathbb{RP}^2, \ \#_{i=1}^{n} T^2.$$

$\Rightarrow$ By showing the associate groups of above are all different, this can lead to a proof of the classification of 2-manifolds that does not involve Euler characteristic, orientability, triangulations, in Topic 7 (Starbird-Su 11.1-11.5, Massey W.S. Ch.1 §5-§7)

For example, Conway's ZIP proof (handle, cross-handle, cap, and cross-cap)

5. One more application is to prove Brouwer Fixed point theorem.

(2d) Let $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$

If $f: D^2 \to D^2$ is continuous, then f has a fixed point.

Non-ex of "$\Leftarrow$" of "Corollary 52.5" 1

$$\pi_1(\mathbb{R}^3 \setminus K_1) \underset{\text{isomorphic}}{\simeq} \pi_1(\mathbb{R}^3 \setminus K_2)$$

$$\not\Rightarrow K_1 \underset{\text{homeomorphic}}{\simeq} K_2. \ (\Leftrightarrow \mathbb{R}^3 \setminus K_1 \simeq \mathbb{R}^3 \setminus K_2)$$

e.g. K_1 = right-handed trefoil knot

K_2 = left-handed trefoil knot

Non-ex of "$\Leftarrow$" of "Cor. 52.5" 2.

$\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^n$, $\{0\}$ (one-pt space)

S^2, are not homeomorphic

but they are all homotopic

$(\pi_1(X_i) = e = 1)$

Part II. Theory (1)

Basic definitions and theorems of fundamental group

Def. A homotopy of two functions

Let $f, g : X \to Y$ be continuous.

Then $f \simeq g$ (f is homotopic to g)

if $\exists H : X \times I \to Y$ such that

$$H(x,0) = H_0 = f(x), \quad \forall x \in X.$$
$$H(x,1) = H_1 = g(x)$$

Then H is called a homotopy between f and g.

Think $\{H(x,\cdot), x \in [0,1]\}$ as a set of continuous functions continuously deformed f to g.

Def. Homotopy equivalence of spaces

Two topological spaces X and Y are homotopy equivalent if

$\exists f : X \to Y$

$\exists g : Y \to X$

such that

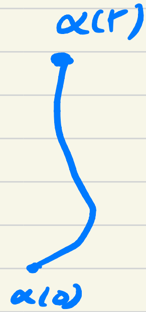
$g \circ f \simeq id_X$

and $f \circ g \simeq Id_Y$.

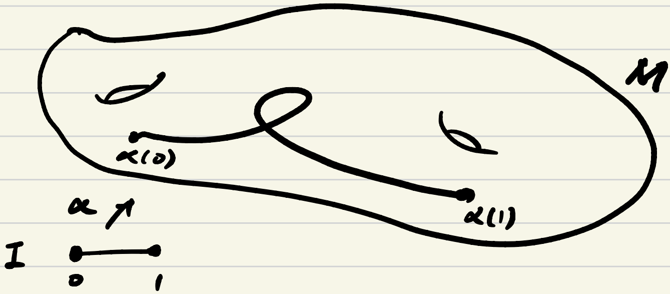
Let X be a topological space.

Def. α is a continuous function that maps a closed interval I to a topological space X, then α is called a path.

$\alpha : [0, r] \to X$, $r \geq 0$, $r \in \mathbb{R}$.



e.g. Let $\alpha : I \to M$



* α can cross or repeat itself.

If α is not normalized,

$W(t) := \alpha(rt)$ is normalized

$\because$ $W(0) = \alpha(0)$, $W(1) = \alpha(r)$, where $\alpha(t) : [0, r] \to X$, $w(t) : [0, 1] \to X$

$\alpha(0)$: initial point
$\alpha(1)$: final point
} endpoints

α is a loop if $\alpha(0)=\alpha(1)$

Def. If $\alpha:[0,r_1]\to X$ and $\alpha(r_1)=\beta(0)$
$\beta:[0,r_2]\to X$

then $\beta*\alpha$ is a product defined as follows

$$(\beta*\alpha)(t):=\begin{cases}\alpha(t), & 0\le t\le r_1\\ \beta(t-r_1), & r_1\le t\le r_1+r_2\end{cases}$$

$\beta*\alpha$ is a path that maps

$[0,r_1+r_2]\to X$.

Def. Path homotopic

Two paths α, β are path homotopic, i.e. $\alpha \simeq_p \beta$ or $\alpha \underset{\cdot}{\sim} \beta$,

if and only if $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$

$\Leftrightarrow$ $\exists$ a homotopy H, $H: A \times I \to Y$, $A \subset X$ { usually take $A = [0,1]$

s.t.

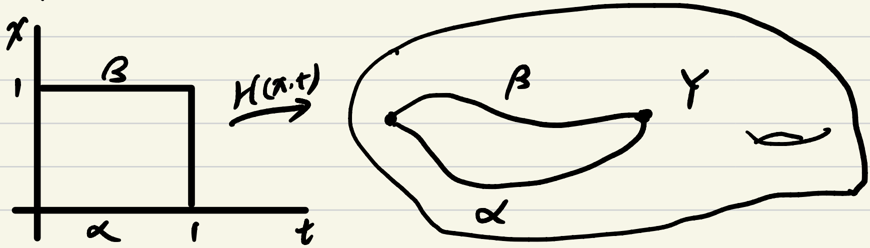
$$H(0,t) = H_0 = \alpha(t)$$

$$H(1,t) = H_1 = \beta(t), \quad \forall t \in [0,1]$$

$$H(x,0) = \alpha(0) = \beta(0)$$

$$H(x,1) = \alpha(1) = \beta(1), \quad \forall x \in [0,1]$$

Eg. (Exercise 12.4 Starbird-Su)



Def **Deformation Retract**

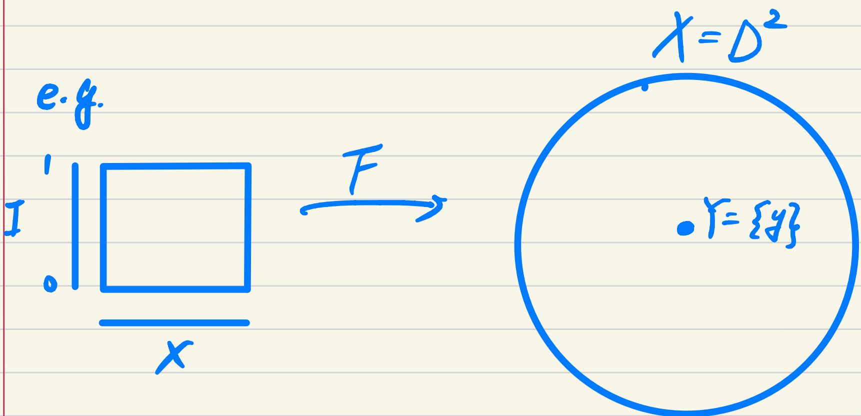
(see also Munkres §52 Exercise 4)

A subspace Y of X is called a deformation retract of X if $\exists$ a homotopy $F: X \times I \to X$

s.t. $\forall x \in X$ and $\forall y \in Y$

1. $F_0(x) = x \in X$
2. $F_1(x) \in Y$
3. $\underset{F(1,y)}{\underset{\shortparallel}{F_1(y)}} = y \in Y$

e.g.



Motivation

Want to find a way to group them (paths) together.

Def. The equivalence classes of paths containing α is denoted by $[\alpha]$.

$$[\alpha] = \{\alpha_i : \alpha_i(0) = \alpha(0),\ \alpha_i(1) = \alpha(1), \text{ and } \alpha_i \sim \alpha\}, \text{ where } i \in I.$$

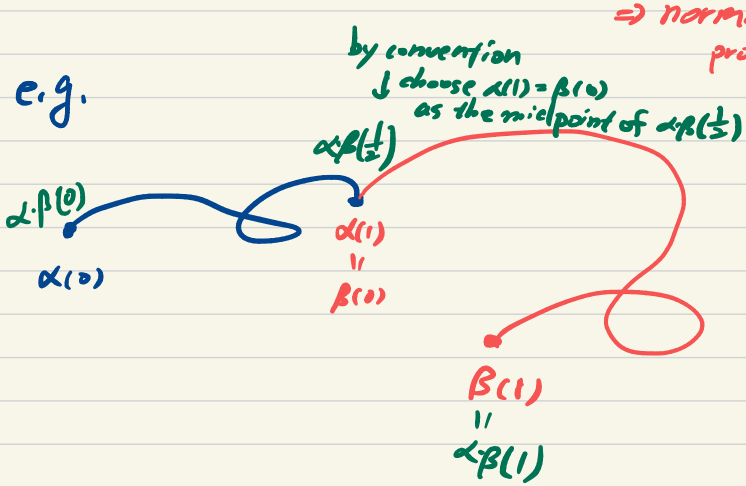
(The operation of the group, $\pi_1(X)$)

Def Let α, β be paths with $\alpha(1) = \beta(0)$
Then their product is denoted as $\alpha \cdot \beta$, meaning that the path first moves along α, followed by moving along β, and it is defined explicitly by

$$\alpha \cdot \beta(x) = \begin{cases} \alpha(2x), & x \in [0, \frac{1}{2}] \\ \beta(2x-1), & x \in [\frac{1}{2}, 1] \end{cases}$$

$\Rightarrow$ normalized product.

e.g.



Extend this def. of a product of paths to the def. of a product of path classes by defining

equivalence class normalized product

$$[\alpha]\cdot[\beta] := [\alpha\cdot\beta]$$

"$\cdot$" is not always well-defined for all homotopy classes unless $\alpha(1)=\beta(0)$

To form a group, all homotopy classes must be referred to a same point x_0 in X called based point.

e.g. $\alpha\cdot\beta\cdot\gamma$

$\Rightarrow \alpha(0)=\alpha(1)=\beta(0)=\beta(1)=\gamma(0)=\gamma(1)=x_0$

Once "$\cdot$" is well-defined, the set of 1^{st}-homotopy classes form a group called

1^{st}-homotopy group, or

Fundamental group, or

Poincaré Group

and denoted as

$\pi_1(X, x_0) = \{[\alpha_i]_{i \in I}$: equivalence classes of homotopic loops based at $x_0 \in X\}$

In general, instead of using loops S^1, we can use S^n, then the n-th homotopy group is denoted as $\pi_n(X)$

$\pi_n(X)$ consists of maps $S^n \to X$ up to a generalization of "path homotopy"

Example 1

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120}\times\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72}\times\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{120}$

For $n \geq 2$, it might be more computable by using homology group H_n.

$\pi_1(X, x_0) = \{[\alpha_i]_{i\in I}$: equivalence classes of homotopic loops based at $x_0 \in X\}$

i.e. → $= \{$closed normalized paths in X based at $x_0\} \Big/ \{$pinted homotopy$\}$

quotient space

* Homotopies are pointed or endpoints are fixed

i.e. $H : \underset{x}{[0,1]} \times \underset{t}{[0,1]} \to X$

s.t. $H(x,0) = H(x,1) = x_0$, $\forall x \in [0,1]$

Lemma 51.1 (Munkres)

Path homotopy is an equivalence relation.

Want to show:

(i) $\alpha \simeq_p \alpha$

(ii) $\alpha \simeq_p \beta \Rightarrow \beta \simeq_p \alpha$

(iii) $\alpha \simeq_p \beta$, $\beta \simeq_p \gamma$

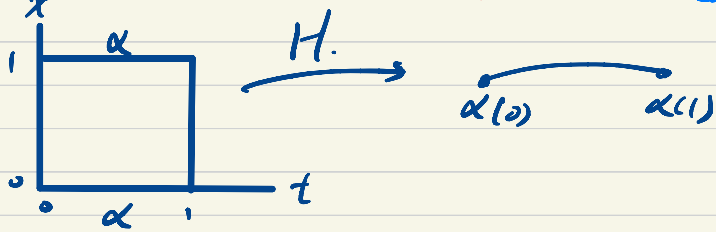
$\Rightarrow \alpha \simeq_p \gamma$

Pf. as examples

(i) Take $H_x(t) = \alpha(t)$

→ what if $\alpha(t) =$ constant?
⇒ $id = e = \alpha_0$ (identity)

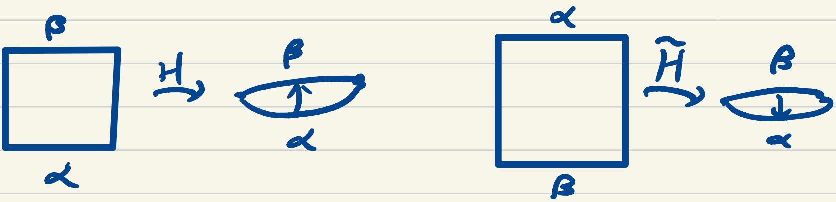
→ what if α goes backward? ⇒ inverse α_{-1}



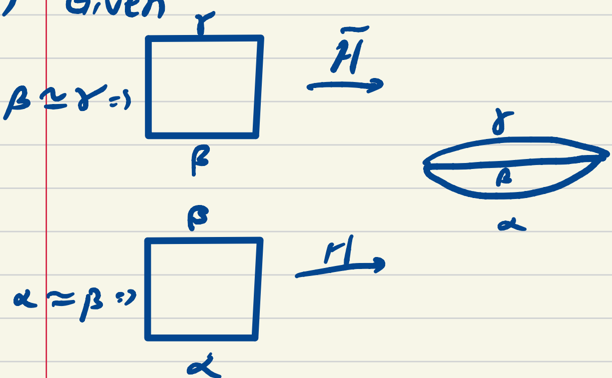
(ii) Given $\alpha \simeq_p \beta$

⇒ $\exists H(x,t)$

Take $\tilde{H}_x(t) = H(1-x, t)$

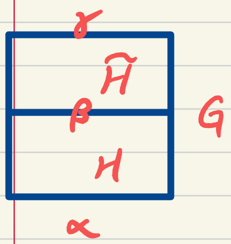


(iii) Given



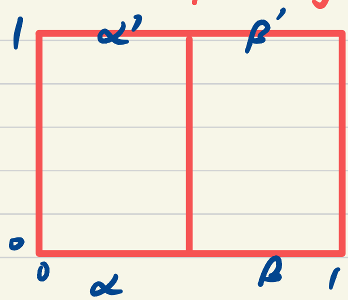
Take

$$G(x,t)=\begin{cases} H(2x,t), & \forall x\in[0,\tfrac{1}{2}] \\ \tilde{H}(2x-1,t), & \forall x\in[\tfrac{1}{2},1] \end{cases}$$



So, we have $id_x=\alpha_0$ and α_{-1},

what if speeding up twice horizontally?



Answer: by using group multiplication we can prove a theorem that can lead to group associativity!

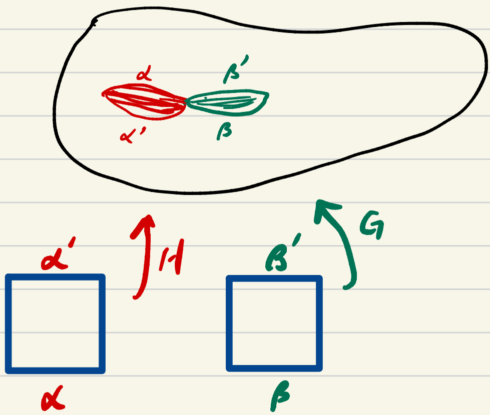
Thm $\alpha, \alpha', \beta, \beta'$ are paths in a space X

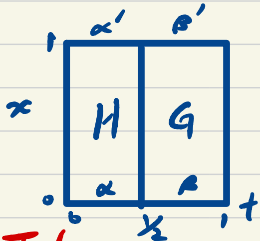
s.t. $\alpha \sim \alpha'$, $\beta \sim \beta'$, and

$\alpha(1) = \beta(0)$, then $\alpha \cdot \beta \sim \alpha' \cdot \beta'$.

Pf.

Given $\alpha \sim \alpha'$ and
$\beta \sim \beta'$





Take

$$F(x,t) = \begin{cases} H(x, 2t) & t \in [0, \tfrac{1}{2}] \\ G(x, 2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

Associativity of $\pi_1(X)$

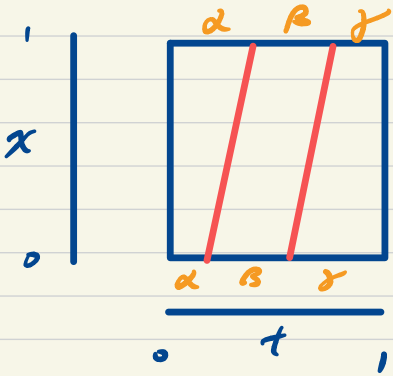
Thm Given paths α, β, and γ where the following products are well-defined

i.e. $\alpha(1)=\beta(0)$, $\beta(1)=\gamma(0)$

Then $(\alpha\cdot\beta)\cdot\gamma \sim \alpha\cdot(\beta\cdot\gamma)$ and

$$([\alpha]\cdot[\beta])\cdot\gamma = [\alpha]\cdot([\beta]\cdot[\gamma])$$

Sketch of Pf.



If a path that doesn't move then it stays at the origin, and then it has a special name.

Def. Let X be a topological space and let x_0 be a point in X.

A map $e_{x_0} : [0,1] \to X$ that ends every point of $[0,1]$ to the single point x_0 is called a constant map or a constant path.

It is also denoted as

id_{x_0} or α_0. (identity of $\pi_1(X)$)

Thm 51.2 (simplified from Munkres)

$(\pi(X, x_0), *)$ is a group. That is

"$*$" has 3 properties as follow.

To simplify, let's apply it to loops.

$\Rightarrow$ 1) $\exists$ identity e

$$[e] = [e_{x_0}] \underset{x_0 = x_1}{=} [e_{x_1}] = [\alpha_0] = \text{constant loop.}$$
$$= [\alpha^0]$$

2) $\exists$ inverse. α^{-1}

i.e. Given a loop α in X,

let α^{-1} denote the path by

$\alpha^{-1}(t) = \alpha(1-t)$

s.t. $[\alpha] * [\alpha^{-1}] = [e] = [\alpha_0]$

3) Associativity

If $[\alpha] * ([\beta] * [\gamma])$ is defined,

so is $([\alpha] * [\beta]) * [\gamma]$, and they are equal.

Thm 51.3 (Munkres)

$*$ It is generalized from associativity.

Let α be a path in X, and let

$r_0, r_1, \ldots, r_n \in [0,1]$ be a partition

of $[0,1]$. $0 \le r_0 < r_1 < \cdots < r_n \le 1$.

Let $\alpha_i : I[0,1] \to X$ be a path, and

α_i is a restricted map of α, i.e.

$$\alpha_i(t) = \alpha(t)\Big|_{t \in [r_{i-1}, r_i]}$$

Then $[\alpha] = [\alpha_1] * \cdots * [\alpha_n]$

Proof.

By induction, based on the proof of associativity.

Part III : Theory (2)

Induced Homomorphism, Isomorphism, and Functorial Properties

$\pi_1(X_i)$ is a topological invariant.

That is if two spaces are homotopic then they have equivalent fundamental group up to isomorphism

Corollary 52.2

X is path connected. $\Leftrightarrow \exists$ a path γ, s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$

$x_0, x_1 \in X$

$\Rightarrow \pi(X, x_0) \underset{\text{isomorphic}}{\simeq} \pi(X, x_1)$

They are only algebraically the same up to isomorphism.

$$[\alpha] \longmapsto [\gamma \cdot \alpha \cdot \gamma^{-1}]$$

$[\gamma^{-1} := \gamma(1-t)$

$\Rightarrow$ $\pi_1(X)$ does not depend on base point x_0, if X is path-connected.

The isomorphism is unique if $\pi(X)$ is abelian.

Let $f:(X,x_0)\to(Y,y_0)$ be a continuous map s.t. $f(x_0)=y_0$

Def. Define a corresponding group homomorphism (induced)

$$f_*:=\Pi_1(f):\pi_1(X,x_0)\to\pi_1(Y,y_0)$$
$$[\alpha]\longmapsto[f\circ\alpha]$$

If $x_0, x_1\in X$, $x_0\neq x_1$ then $(h_{x_0})_*\neq(h_{x_1})_*$.

Pf.

Since $(h_{x_0})_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)$.

$(h_{x_1})_*:\pi_1(X,x_1)\to\pi_1(Y,y_1)$

Since two domains $\pi_1(X,x_0)\neq\pi(X,x_1)$.

$\Rightarrow$ if $x_0=x_1$, only one base point under consideration then $(h_{x_0})_*=(h_{x_1})_*:=h_*$

So, although if X is path connected, $\pi_1(X)$ does not dep. on x_0, $(h_{x_0})_*$ is still depending on x_0.

There are numerous mathematical notions (objects) and maps that are structure preserved (morphisms).

Objects:

sets, groups, rings, modules, vector spaces, topological spaces, smooth manifold, partially ordered sets

Morphisms:

functions, group homomorphisms, ring homomorphisms, module homomorphisms, linear maps, continuous maps, smooth maps, ordered-preserving functions

The structure-preserving maps from one category (object) to another are called functors.

In algebric topology, the map (induced homomorphism) is an example. The fundametal group functors map from the category of (pointed) topological spaces to the category of groups, i.e. $\pi_1 : Top^* \longrightarrow Grp$

where Top^* denotes the category of pointed topological spaces, and Grp stands for the category of group. $\pi_1 : Top^* \to Grp$ consists of two data:

(i) an objects functor

$\pi_1 : (X, x_0) \longmapsto \pi_1(X, x_0)$

(ii) a morphism functor

$\pi_1(f) = f_* \quad f \longmapsto f_*$

i.e. π_1 also sends a continuous map f between topo. spaces to a homomorphism between the corresponding fundamental groups.

Functorial properties

Thm 52.4
Corollary 52.5 } in Munkres. or { Thm 12.24 in Starbird and Su }

→ If $X \simeq Y \Rightarrow f_*$ is a Group isomorphism.

→ Thm 52.4 means f_* is a group homomorphism.

capitalized

$\Pi_1 : C_1 \to C_2$ where $C_1 = Top^*$, $C_2 = Grp$ $\because \Pi_1(X, x_0) = \Pi_1(X) = \pi_1(X)$

s.t. { ① $Ob(C_1) \mapsto Ob(C_2)$
② $Hom(X, Y) \to Hom(\Pi_1(X), \Pi_1(Y)) = Hom(\pi_1(X), \pi_1(Y))$ }

Thm 12.24 in Starbird & Su {
$\Pi_1(id_X) = id_{\pi_1 X}$

$f_*, g_* \in Hom(\pi_1(X), \pi_1(Y))$

$\Pi_1(f \circ g) = \Pi_1(f) \circ \Pi_1(g) = f_* \circ g_*$ } Thm 52.4 in Munkres

$f, g \in Hom(X, Y)$

$\because \Pi_1(f) = f_*$

induced homomorphism

To show Π_1 is actually a functor!

Thm 52.4 Group Homomorphism

If $\begin{cases} h : (X, x_0) \longrightarrow (Y, y_0) \\ k : (Y, y_0) \longrightarrow (Z, z_0) \end{cases}$ are continuous

then $(k \circ h)_* = k_* \circ h_*$

Pf. By def. $g_*([\alpha]) = [g \circ \alpha]$

$$lhs = (k \circ h)_* ([\alpha]) = [(k \circ h) \circ \alpha]$$

$$\begin{aligned} rhs = (k_* \circ h_*)([\alpha]) &= k_*(h_*([\alpha])) \\ &= k_*([h \circ \alpha]) \\ &= [k \circ (h \circ \alpha)] \\ &= [(k \circ h) \circ \alpha] = lhs. \end{aligned}$$

Corollary 52.5 Group Isomorphism

Let $h : (X, x_0) \to (Y, y_0)$ be a homeomorphism.

$\Rightarrow h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

i.e. $\pi_1(X) \simeq \pi_1(Y)$

Pf:

Let $k : (Y, y_0) \to (X, x_0)$, $k = h^{-1}$.

$$\Rightarrow \begin{cases} k_* \circ h_* = (k \circ h)_* = (h^{-1} \circ h)_* = i_* \\ h_* \circ k_* = (h \circ k)_* = (h \circ h^{-1})_* = j_* \end{cases}$$

(where $\begin{cases} i \text{ is an identity map of } (X, x_0) \\ j \text{ is an identity map of } (Y, y_0) \end{cases}$)

$\Rightarrow i_*, j_*$ are identity group isomorphisms of $\pi_1(X, x_0)$, $\pi_1(Y, y_0)$

$\Rightarrow k_* = h_*^{-1}$

Recall: $\Pi_1(h) = h_*$

if X, Y are path-connected

$\Pi_1 : (X, x_0) \mapsto \pi_1(X, x_0) = \pi_1(X)$

$\Pi_1 : (Y, y_0) \mapsto \pi_1(Y, y_0) = \pi_1(Y)$

In general, suppose C_1 and C_2 are two categories. (e.g. group and topological space)

Def. A functor $F: C_1 \to C_2$ consists of the following data:

① an object functor (a class-function)

$$F: ob(C_1) \to ob(C_2)$$
$$c \mapsto Fc$$

② a morphism functor (a function on Hom-sets)

$\forall a, b \in C_1$, $\exists$ a function F on $Hom(a,b)$ such that

$$F: Hom(a,b) \longrightarrow Hom(Fa, Fb)$$

$Fa \in C_2$, $Fb \in C_2$

$f \in Hom(a,b)$

$$f \longmapsto Ff$$

If $f: a \to b$, $g: b \to c$ then we require $F(g \circ f) = F(g) \circ F(f)$, and $\forall c \in C_1$ $F(id_c) = id_{Fc}$.

(Cartesian Product)

12.20 in Starbird–Su

Thm $\pi_1(X \times Y, (x_0, y_0))$

$$\simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

$$[\alpha] \longrightarrow ([p \cdot \alpha], [q \cdot \alpha])$$

$$\begin{array}{ccc} X & \times & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & & Y \end{array}$$

projections

exercise 12.21 in Starbird & Su

Corollary:

$$\pi_1(T) \cong \pi_1(S^1 \times S^1)$$

$$= \pi_1(S^1) \times \pi_1(S^1)$$

$$\simeq \mathbb{Z} \times \mathbb{Z}.$$

e.g. $\pi_1(X)$, X is a solid torus $D^2 \times S^1$

$$1 \times \mathbb{Z}$$

$$\pi_1(S^1 \times S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

Seifert – Van Kampen

Given : $X = U \cup V$, U and V are open, X is path-connected.

Know : $\pi_1(U)$, $\pi_1(V)$, and $\pi_1(U \cap V)$.

Find $\pi_1(U \cup V)$

a set of generators of A_1

Pushout diagram.

G is a pushout. Let $A_1 = \langle \Phi_1 : R \rangle$

a set of relations

if $\exists A_1 \xrightarrow{f_*} G'$ $\quad A_2 = \langle \Phi_2 : S \rangle$

$\exists A_2 \xrightarrow{g_*} G'$ $\quad A_0$ is generated by Ψ.

with $f_* \beta_* = g_* \alpha_*$, then there is a unique map $G \to G'$ s.t. the diagram commute $\beta'_* \alpha_* = \alpha'_* \beta_*$

$G = \langle \Phi_1 \cup \Phi_2 : R \cup S \cup \{\alpha(z)\beta^{-1}(z) : z \in \Psi\} \rangle$

$$\begin{tikzcd}
A_0 \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"'] & A_1 \arrow[d, "\beta'_*"] \arrow[ddr, bend left, "g_*"] & \\
A_2 \arrow[r, "\alpha'_*"] \arrow[drr, bend right, "f_*"'] & G \arrow[dr] & \\
 & & G'
\end{tikzcd}$$

Thm 12.53 (Starbird-Su)

Seifert-Van Kampen group presentation version

Let $X = U \cup V$, U and V are open and path-connected and $U \cap V$ is path-connected and $x \in U \cap V$

Let $\left.\begin{array}{l} i : U \cap V \to U \\ j : U \cap V \to V \end{array}\right.$ be the inclusion maps.

Let $\underbrace{\pi_1(U,x)}_{= A_1} = \langle \underbrace{g_1, \dots, g_n}_{\in \Phi_1} \mid \underbrace{r_1, \dots, r_m}_{\in R} \rangle$

$\underbrace{\pi_1(V,x)}_{= A_2} = \langle \underbrace{h_1, \dots, h_t}_{\in \Phi_2} \mid \underbrace{s_1, \dots, s_u}_{\in S} \rangle$

$\underbrace{\pi_1(U \cap V, x)}_{= A_0} = \langle k_1, \dots, k_v \mid t_1, \dots, t_w \rangle$

Then

$$\overbrace{\pi_1(X,x)}^{G =} = \langle \overbrace{g_1, \dots, g_n, h_1, \dots, h_t}^{\Phi_1 \cup \Phi_2} \mid \underbrace{r_1, \dots, r_m, s_1, \dots, s_u, \underset{\substack{\uparrow \\ \text{induced homomorphism}}}{i_*}\underbrace{(k_1) j_*(k_1^{-1})}_{\text{conjugate}}, \dots, i_*(k_v) j_*(k_v^{-1})}_{R \cup S \cup \alpha(z)\beta^{-1}(z),\ z \in \Phi} \rangle$$

Part IV: Examples of Computations of Fundamental Group

Techniques to compute $\pi_1(X)$:

1. based on specific geometry of the space, e.g. S^1, D^2, and many simply connected spaces.

2. writing the space into a Cartesian product of spaces whose fundamental group we already know.

3. showing the space is homotopy equivalent to a space whose fundamental group we know

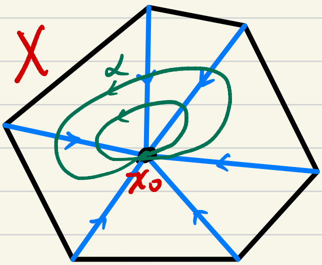
4. find a deformation retraction to a space, e.g. $D^2 \simeq \{0\}$, whose fundamental group we know.

5. Using Steifert – Van Kampen Thm.

6. Using Covering spaces and homotopy lifting theorems in (Munkres, Starbird-Su, and Hatcher).

Example 1.

Given X as a polygon, i.e. a disk D^2.



All loops at x_0 are nul-homotopic, i.e., homotopic to the constant loop at x_0 (i.e. a constant loop at x_0 means it just sits there and do nothing in the movie.)

This retraction deformation of the entire space to an one-point space, hence every point in this space is identical to x_0.

Also, for the one-point space, $\exists!$ element in the fundamental group

$$\pi_1(X, x_0) \overset{\text{for simplicity}}{=} \pi(X) = \pi(\text{disk}) = \pi(\text{polygon})$$
$$= \pi(D^2) = \{[\alpha_0]\} = \{[e]\} \underset{\text{Starbird-Su}}{=} 1 \left(\underset{\text{Munkres}}{\overset{\text{or}}{=}} 0\right)$$

$\forall u \in \pi_1(X, x_0)$, $u \sim e_{x_0}$

Def. u is called nulhomotopic.

Note that however, if two space are associated to the same fundamental group, they can be non-homeomorphic.

Exercise 12.13 in Starbird & Su

e.g. 1. $\mathbb{R}$

2. $\mathbb{R}^2$, $\mathbb{R}^n$

3. $\{0\}$ one point space

none of them are homeomorphic

Thm 12.15 → 4. S^2 5. $[0,1] \subset \mathbb{R}$

they are all $\pi_1(X_i) = e = 1$

homotopic up to isomorphism

↑ notation in Starbird-Su

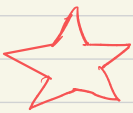
They are all similar examples of

Example 1.

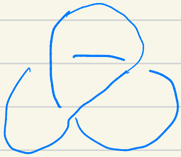
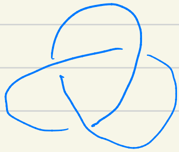
More examples: $\pi_1(X) = 1$

X = a cone, X = a convex set in $\mathbb{R}^n$

X = a star-like space in $\mathbb{R}^n$



One more similar example of example 1.

K_1  and  K_2

associate to the same fundamental group $\pi_1(\mathbb{R}^3 - K_1) = \pi_1(\mathbb{R}^3 - K_2)$ but K_1 and K_2 are not isotopic.

$$\pi_1(\mathbb{R}^3 - K_1) = \pi_1(\mathbb{R}^3 - K_2)$$
$$= \langle x, y \mid y^3 = x^2 \rangle$$

* Observation:

Knot Group $G(K)$

$:= \pi_1(\mathbb{R}^3 \setminus K)$

elements in $\pi_1(\mathbb{R}^3 \setminus K)$ are equivalence classes of loops based on ambient space $\mathbb{R}^3 \setminus K$

$\Rightarrow$ Since the space is the complement of the knot, the elements of $\pi_1(\mathbb{R}^3 \setminus K)$ are homotopic classes of loops which do not intersect the knot.

$\Rightarrow$ If a loop α does not intersect K, then $\alpha \subset \mathbb{R}^3 \setminus K$ (by right-hand rule)

Def A knot is an embedding (1 to 1) from S^1 to $\mathbb{R}^3$.

$$f : S^1 \rightarrowtail \mathbb{R}^3$$

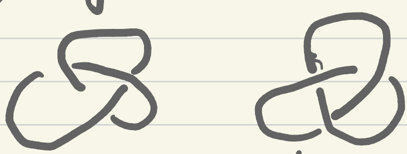
Def Two knots are isotopic if $\exists$ an isotopy $f_t(x)$, $t \in [0,1]$ s.t.

$$f = f_0 \text{ and } g = f_1.$$

Def Knot diagrams, e.g.

it's a path in $\mathbb{R}^3$, not a space.

unknot



trefoil knots

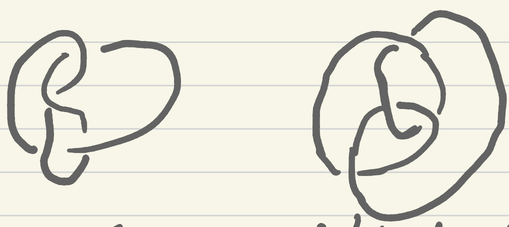


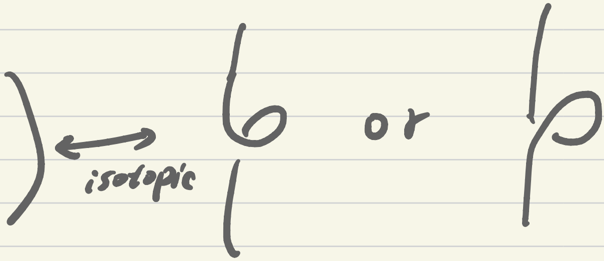
figure-eight knots

* Any diagram becomes an unknot if we change some crossing from

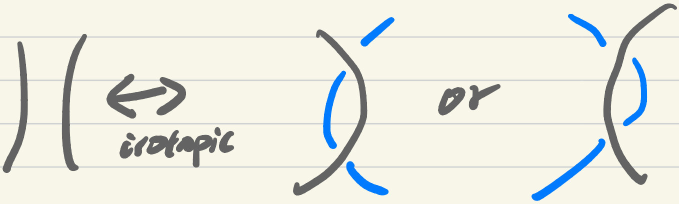
 to 

Def Reidmeister Moves

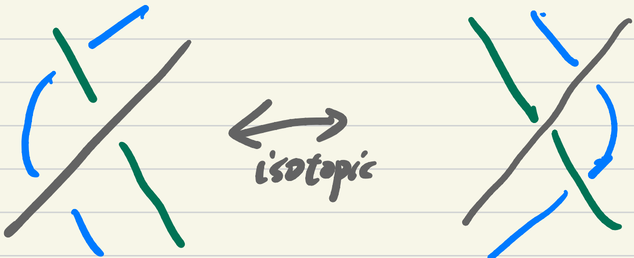
R1



R2

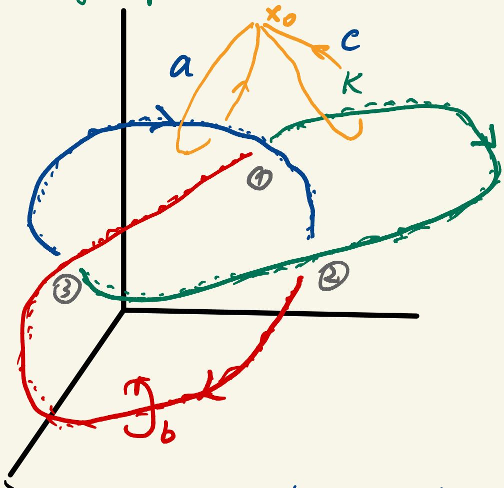


R3



only move the blue strand

The group of trefoil knot $\pi_1(\mathbb{R}^3 \setminus K)$ is the fundamental group of the knot's complement in space.



Fundamental Quandle

Arcs

$x \quad y \quad z$

Crossing relation

① $z = y \triangleright x$

② $y = x \triangleright z$

③ $x = z \triangleright y$

Knot Group

Loops

$\langle a, b, c$

Conjugation relations

$ac = ba \Rightarrow c = a^{-1}ba,$

$b = c^{-1}ac,$

$a = b^{-1}cb \rangle$

Wirtinger presentation derived by using Reidmeister moves.

Left and right handed trefoil knots have the same presentation!

Simplifying the previous result:

$$\pi_1(\mathbb{R}^3 \setminus K) = \langle a, b, c \mid c = a^{-1}ba,\ b = c^{-1}ac,\ a = b^{-1}cb \rangle$$

$$\Rightarrow a = b^{-1}(a^{-1}ba)b \quad \text{— ①}$$

$$b = (a^{-1}ba)^{-1}a(a^{-1}ba) \quad \text{— ②}$$

$$= a^{-1}b^{-1}aaa^{-1}ba$$

$$= a^{-1}b^{-1}aba$$

$$\Rightarrow ab = b^{-1}aba$$

$$\Rightarrow bab = aba$$

Or, if start with ②:

$$aba = bab$$

$$\Rightarrow \boxed{\pi_1(\mathbb{R}^3 \setminus K) = \langle a, b \mid aba = bab \rangle}$$

$$aba\ aba = aba\ bab$$

$$\Rightarrow \underbrace{bab}_{aba}\ aba = aba\ bab$$

$$\Rightarrow baabaa = ababab \qquad \text{Let } x = ab,\ y = ba^2$$

$$\Rightarrow y^2 = x^3 \Leftrightarrow \langle x, y \mid y^2 = x^3 \rangle, \quad a = x^{-1}y, \quad b = y^{-1}x^2$$

Some more details:

$$a = \underbrace{x\, b^{-1}}$$

$$b = y\, a^{-2}$$

$$= y\, b^2 x^{-2}$$

$$y^{-1} b = b^2 x^{-2}$$

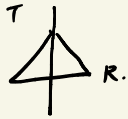
$$\Rightarrow b\, y^{-1} = b^2 x^{-2}$$

$$y^{-1} = b\, x^{-2}$$

$$b = y^{-1} x^2$$

$$a = x\, x^{-2} y = x^{-1} y$$

Dihedral group D_6.



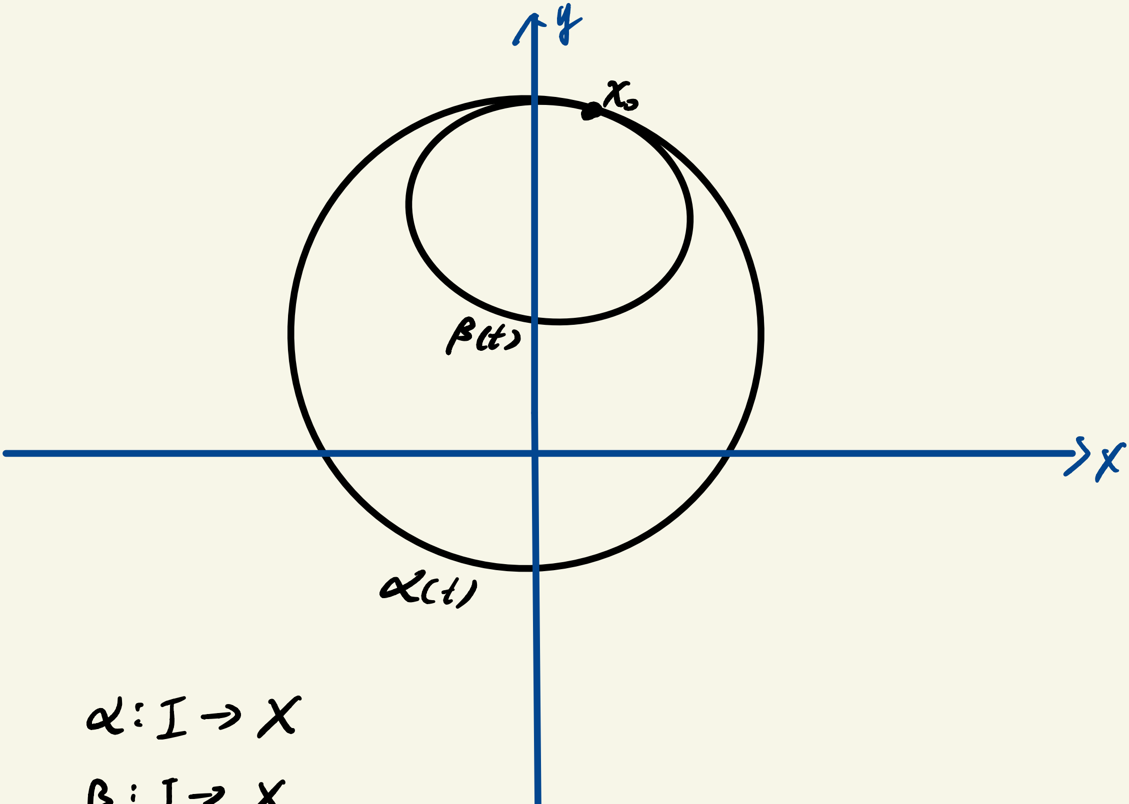
$$D_6 = \langle R, T \mid R^3 = T^2 = e = RTRT \rangle$$

the simplified presentation of a trefoil is similar (large actually since it's infinite) to D_6 (finite) group.

$$\underbrace{\phi}_{\text{surjective homomorphism}} : \underbrace{G(K)}_{\text{larger than } D_6} \twoheadrightarrow D_6$$

Since all knots are homeomorphic to S^1, so don't be confused with $\pi_1(S^1) = \mathbb{Z}$, and an unknot has a shape like S^1 and say $\pi_1(\text{unknot}) = \mathbb{Z}$. The way to use fundamental group in distinguish knots is by considering the topological structure of the complement of a knot and by taking a based point in $\mathbb{R}^3 \setminus K$, and use loops based at that point to investigate the space $\mathbb{R}^3 \setminus K$. A more systematic way is using braid and solid torus (in the appendix).

Ambient space $X = \mathbb{R}^2$



$\alpha : I \to X$

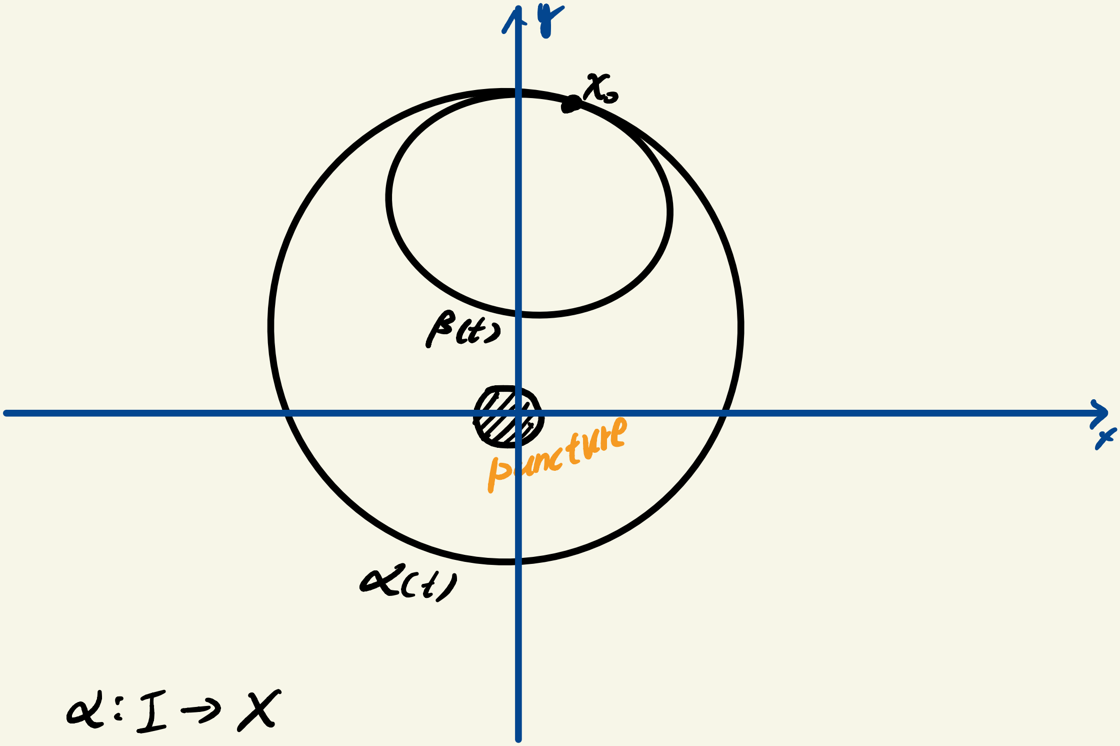
$\beta : I \to X$

$\exists H_x : [0,1] \to X$

$H_x(t) = (1-x)\,\alpha(t) + x\,\beta(t)\,, \quad x \in [0,1]$

$\pi_1(X) = \{\alpha_0\} = 1$

Ambient space $X = \mathbb{R}^2 \setminus \{0,0\}$



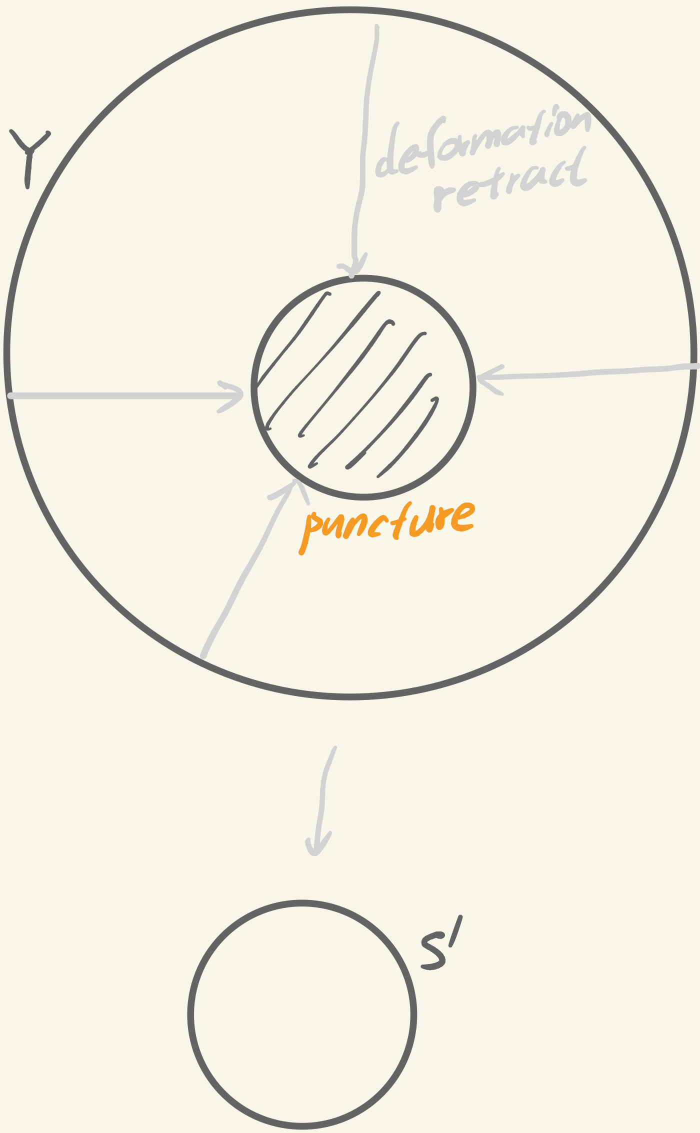
$\alpha : I \to X$

$\beta : I \to X$

$\cancel{\exists} H_x : I[0,1] \to X$

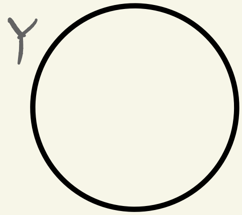
$\pi_1(X) = \{\alpha_0, \alpha_1\} = \langle \alpha_1 \rangle \simeq \mathbb{Z}$

$X = \mathbb{R}^2 \setminus \{0,0\}$ Consider a subspace topology $Y \subseteq X$



Example 2.

$Y := S^1$

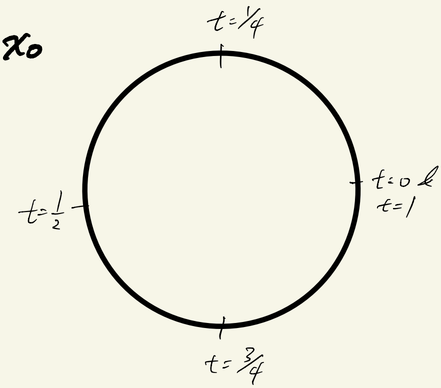


$\alpha_0 =$ constant loop $= x_0$

$\alpha_1(t) = t \pmod 1$

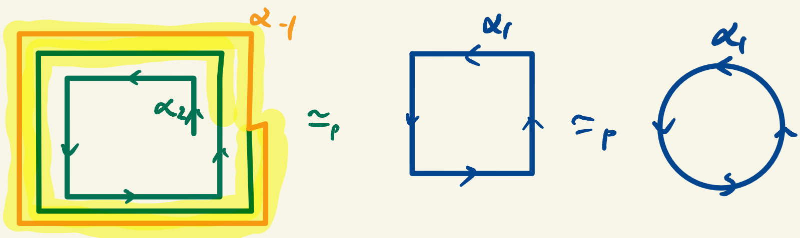
$\alpha_n(t) = (n+1)t \pmod 1$

$n \in \mathbb{Z}$, $t \in \mathbb{R}$.



$\alpha_{-1}(t) = -t \pmod 1$

$[\alpha_2] * [\alpha_{-1}] = [\alpha_1]$



$\pi_1(S^1) \cong \mathbb{Z}$

C.f. $\pi_1(S^2) \simeq e \simeq \pi_1(\mathbb{R}^1) \simeq \pi_1(\mathbb{R}^2)$

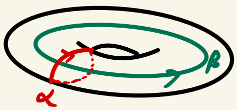
$\simeq \cdots \simeq \pi_1(\mathbb{R}^n)$

all closed loops, S^1, based at x_0 on S^2 can shrink by using a homotopy to their based point x_0

Example 3. $(S^1 \times S^1)$

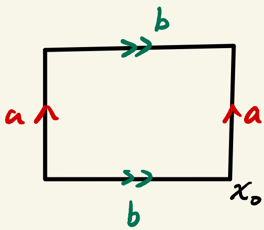
Given $X = T^2$, a torus 2d surface

X is a topological space.



Claim: α, β are two only generator of $\pi(T^2)$.

S^1



read G: $aba^{-1}b^{-1}$

notice the loops α, β are on the edges a, b

$\Rightarrow [\alpha] * [\beta] * [\alpha^{-1}] * [\beta^{-1}]$

$= [e]$

$\Rightarrow [\alpha] * [\beta] = [\beta] * [\alpha]$

$\Rightarrow \pi(T^2)$ is commute.

e.g. $[\alpha^5] * [\beta^6] * [\alpha^{-3}] * [\beta^{-7}]$

$\underset{T}{=} [\alpha^5] * [\alpha^{-3}] * [\beta^6] * [\beta^{-7}]$

$\because \pi(T^2)$ is commutative

$= [\alpha^2] * [\beta^{-1}]$

$\Rightarrow$ in general, if $y \in \pi(T^2)$ then y has the form $[\alpha^m] * [\beta^n]$

$m, n \in \mathbb{Z}$.

$\Rightarrow \pi(T^2) = \mathbb{Z} \times \mathbb{Z}$

Example 4 M_g a g-genus torus, $g \geq 2$.

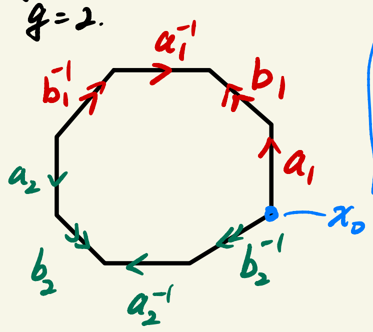
Claim: $\pi(M_g)$ is non-abelian.

A genus g surface (aka g-torus) is a surface formed by connected sum of g-many tori.

$g=1$

---

e.g. $g=2$.



$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1} = e$

$\Rightarrow [\alpha_1, \beta_1][\alpha_2, \beta_2] = e$

$[\alpha_i, \beta_i] := \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$

which cannot be simplified to $\alpha_i \beta_i = \beta_i \alpha_i$

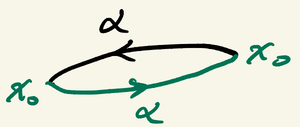
Additionally,

$\pi(M_2) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] = e \rangle$

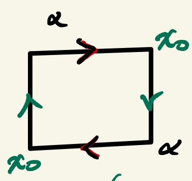
Example 5

Real projective plane.

identifying antipodal $x \in \partial D^2$
$x \sim -x$



or



$\pi(RP_2) = \{[e], [\alpha]\} \cong (\mathbb{Z}_2, +) = \{0, 1\}$ (w.r.t. + addition group)

or $\simeq (\mathbb{Z}_2, \cdot) = \{-1, +1\}$ wrt "$\cdot$" multiplication group.

* $\pi(RP_2)$ is a torsion group since $[\alpha^m] = [e]$

Example 6.

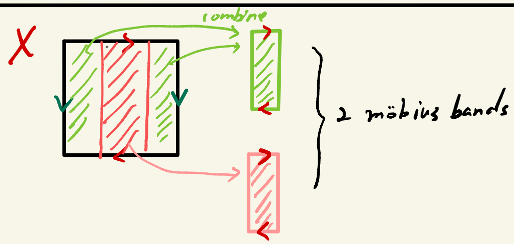
π(Möbius band) = ?

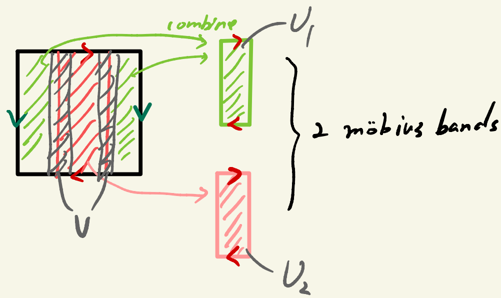
By deformation retraction on to its core circle $\Rightarrow \pi$(Möbius band)

$[0,1]\times\{\frac{1}{2}\}$ $\quad \simeq \pi(S^1) = \langle 1 \rangle \simeq \mathbb{Z}$

Example 7.

π(Klein bottle) = ?





$U_1 \cap U_2 = V$, V is homeomorphic to a cylinder, π(cylinder) $\simeq \pi(S^1)$ deformation retract
$\simeq \mathbb{Z}$.
$\simeq \langle a \rangle$

$\pi(U_1) \simeq \langle x \rangle \simeq \mathbb{Z}$, $\pi(U_2) \simeq \langle y \rangle \simeq \mathbb{Z}$

By Seifert-Van Kampen

$$\pi(X) = (\mathbb{Z} * \mathbb{Z})/N$$

N is normal subgroup generated by the element $(i_{U_1})_*(a)(i_{U_2})_*(a^{-1})$

$\Rightarrow \pi(X) = \langle x, y \mid x^2 = y^2 \rangle$.

Conclusion:

By using Cor 52.5 Functorial Property (its contrapositive), we can classify the 47 calculating examples into the following 10 fundamental groups and use Corollary 52.5 to show that if two spaces are associating to different groups then they are not homeomorphic.

(1) $\pi_1(X) = 1$

X can be: $[0,1]$, $\mathbb{R}^n$, S^2, $\{0\}$, D^2, a cone, a convex set in $\mathbb{R}^n$, a star-like space in $\mathbb{R}^n$, spaces in a shape such as: X, Y, Z, T, S, C, E, F, G, H, I, J, K, L, M, N, U, V, W.

(2) $\pi_1(X) = \mathbb{Z}$.

X can be: S^1, $\mathbb{R}^3 \setminus K$ where K is an unknot, $\mathbb{R}^2 \setminus \{0,0\}$, a Möbius band, a cylinder, A, D, O, P, Q, R-shape spaces.

(3) $\pi_1(X) = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n\text{-many}}$

X can be $\underbrace{S^1 \times \cdots \times S^1}_{n\text{-many}}$

If $n=2$, $S^1 \times S^1$. Or, X can be n-leafed rose

$\Rightarrow$ figure-eight or a torus

(4) $\pi_1(X) = (\mathbb{Z}_2, +)$

if $X = RP_2$

(5) $\pi_1(X) = (\mathbb{Z}_2, \cdot)$

if $X = RP_2$

(6) $\pi_1(X) = \langle x, y \mid x^2 = y^2 \rangle$

if X = a Klein bottle

(7) $\pi_1(X) = \langle x, y \mid x^3 = y^2 \rangle$

if X = a trefoil knot

(8) $\pi_1(X) = \langle \alpha_1, \alpha_2 \mid \alpha_1\alpha_2\alpha_1 = \alpha_2\alpha_1\alpha_2 \rangle$

if X = a trefoil knot

(9) $\pi_1(X) = \langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_1 = \alpha_3\alpha_2\alpha_3^{-1},$

$\alpha_2 = \alpha_1\alpha_3\alpha_1^{-1},$

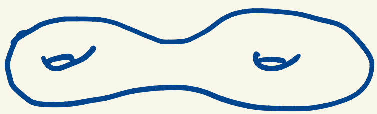
$\alpha_3 = \alpha_2\alpha_1\alpha_2^{-1} \rangle$

if X = a trefoil knot

(10) $\pi_1(X) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid$

$[\alpha_1, \beta_1] \cdot [\alpha_2, \beta_2] = \underset{\substack{\shortparallel \\ id}}{e} \rangle$

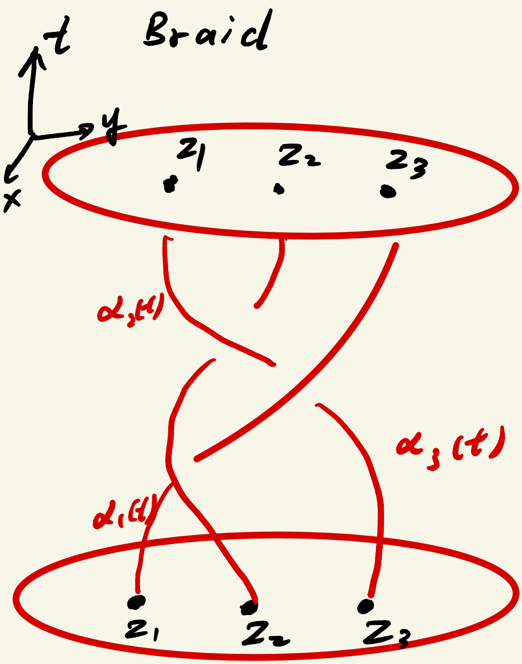
where $[\alpha_i, \beta_i] := \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$

if $X = M_2$ 

Appendix: A systematic way to compute knot group by using braid group and Seifert-Van Kampan Thm.

Q: How to write down a fundamental group of a complement of a knot, i.e. a group of a knot?

Consider three paths $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{R}^3$ as follows:



$t \in [0,1]$

$\alpha_i(t) = (F_i(t), t)$

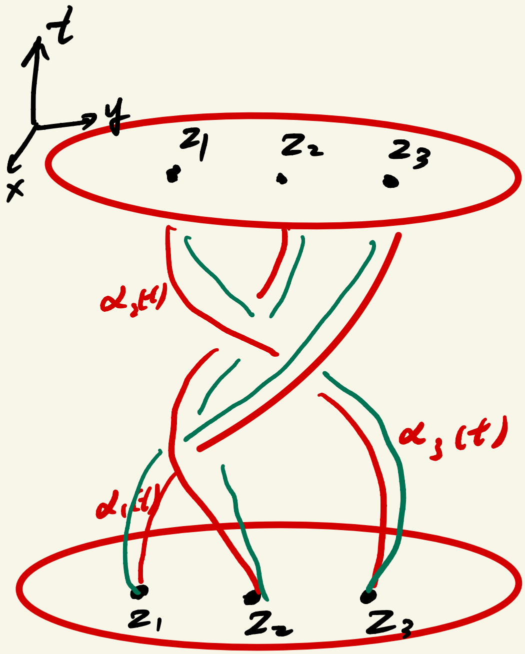
$F_i(t) \neq F_j(t)$, $i \neq j$, $\forall t$.

i.e. not allow X

and each t, $\exists!$ 1 crossing.

$F_i(1) = Z_{s(i)}$ ← permutation

$F_i(0) = Z_i$



braids

$(F_i(t), t)$ and $(G_i(t), t)$ — they are equivalent if

$\exists\, H_i(s,t)$, H_i is a homotopy.

s.t. $H_i(0,t) = F_i(t)$

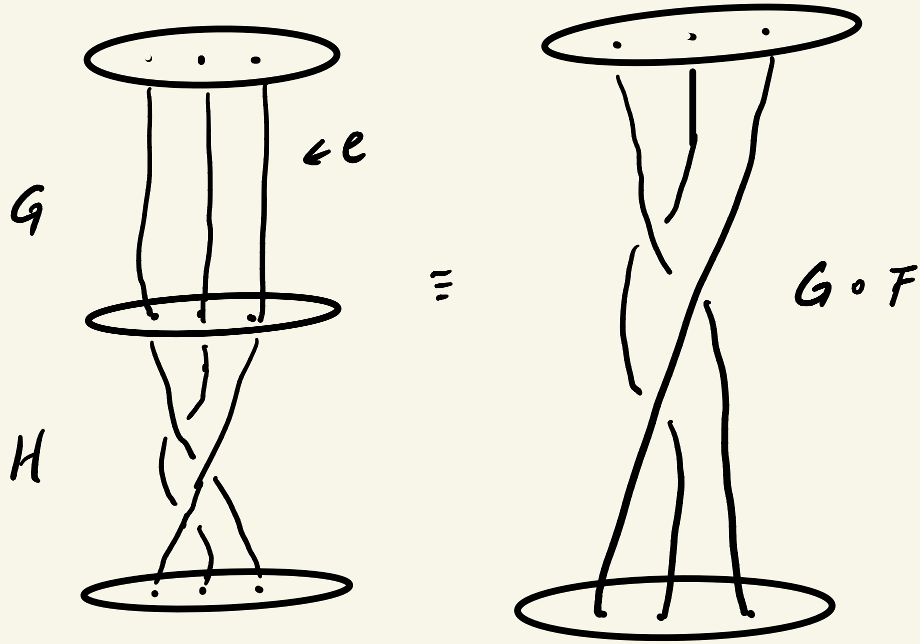
$H_i(1,t) = G_i(t)$

i.e. for each s, H_i defines a braid.

Then we can derive a new group (braid group) law.

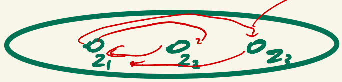
e.g.

(i) e
(ii) inverse
(iii) associativity.



→ configuration space

points are moving around.



for each cross-section

$OC_k = \{(z_1, \dots, z_k) : z_i \neq z_j \text{ if } i \neq j\}$

$UC_k = OC_k / \underbrace{S_k}_{\text{permutation}}$

$\Rightarrow$ Braid group $= \pi_1(UC_k(z_1, \dots, z_k))$.

Generators of Braid group.

 σ_1

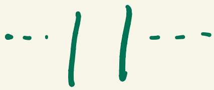
 σ_2

$\vdots$

---

some properties for computations:

(commutativity)

σ_i  σ_j 

σ_j  σ_i 

$\sigma_i \sigma_j$ $\qquad$ $\sigma_j \sigma_i$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ if } |i-j| \geq 2.$$

 σ_i

 σ_{i+1}

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

Artin Action

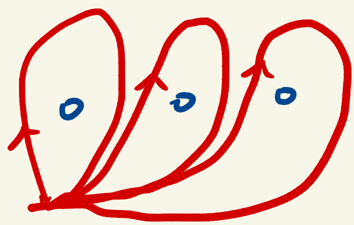
An Artin Action is an action of the n-strand braid group B_n by automorphisms on the free group F_n.

$B_n = \pi_1(\underset{\uparrow}{UC_k}, Z)$ ⟵ Z is a collection of points $z_1, z_2, \dots, z_n$

unordered configuration space on n points

$F_n = \pi_1(\underbrace{\mathbb{R}^2 \setminus Z})$

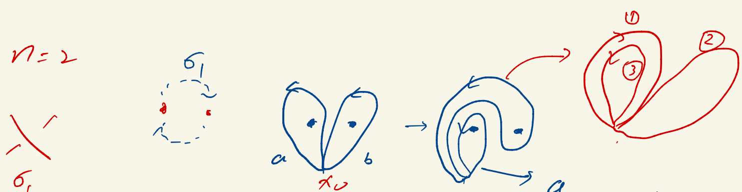
e.g. punchered 3 pts on $\mathbb{R}^2$



$F_3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ from 3 free groups,

3 generators.

$n = 2$



aba^{-1}

$F_2 = \langle a, b \rangle = \mathbb{Z} \times \mathbb{Z}$

$\Rightarrow \sigma_1 : \begin{array}{l} a \mapsto aba^{-1} \\ b \mapsto a \end{array} \Leftrightarrow \begin{array}{l} \phi(a) = aba^{-1} \\ \phi(b) = a \end{array}$

Def Suppose B is an n-strand braid inside $D^2 \times [0,1]$.

Consider the quotient space

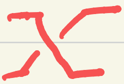
$$D^2 \times S^1 = (D^2 \times [0,1])/\sim, \quad (x,0) \sim (x,1)$$

$\Rightarrow$ B closes up to become a collection of embeded circles $\underbrace{C_B}_{\text{a knot}}$ in $D^2 \times S^1$

$\Rightarrow$ It is called the braid closure C_B of B.

e.g. 2-strand B $\sigma_1{}'$ is an unknot

B:



is a braid

$\rightarrow$ C_B = an unknot

e.g. 2-strand B σ_1^3 is a trefoil

B:

$\rightarrow$ C_B = a trefoil

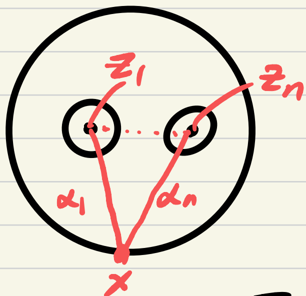
Lemma Suppose $X_B = (D^2 \times S^1) \setminus C_B$ is the complement of $C_B \subset D^2 \times S^1$ (a knot)

Let $x = [\{1\} \times \{0\}] \in (D^2 \times [0,1])/\sim$.
$\hookrightarrow [1, 0]$ $\quad D^2 \subset \mathbb{C} \Rightarrow 1 \in D^2$

$$\pi_1(X_B, x) = \langle \alpha_1, \ldots, \alpha_n, \lambda \mid g \alpha_k g^{-1} = \phi(\alpha_k), \ k = 1, \ldots, n \rangle$$

λ is the loop $x \times S^1$

For $k \in \{1, \ldots, n\}$, α_k is the element of $\pi_1(D^2 \setminus \{z_1, \ldots, z_n\})$ given by the loop in the below figure and



$\phi(\alpha_k)$ denotes the Artin action of B on $\underbrace{\alpha_k \in \pi_1(D^2 \setminus \{z_1, \ldots, z_n\})}$

Thm Let $D^2 \times S^1$ be embeded as the standard solid torus in $\mathbb{R}^3$.

$\Rightarrow \mathbb{R}^3 \setminus C_B$ has the fundamental group:

$$\pi_1(\mathbb{R}^3 \setminus C_B) = \langle \alpha_1, \dots, \alpha_k \mid \alpha_k = \phi(\alpha_k), \\ k = 1, \dots, n \rangle.$$

where $\phi(\alpha_k)$ is the Artin action of B on the free group $\langle \alpha_1, \dots, \alpha_k \rangle$.

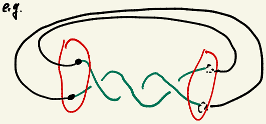
Idea. Take $\lambda = 1$ in the lemma.

(Attaching 2-CW-cells along the circle $x \times S^1$ such that the relation $\lambda = 1$ in the presentation from the lemma)

Q: How to write down a fundamental group of a complement of a knot, i.e. a group of a knot?

A: idea 1. any knot can be put into a braid

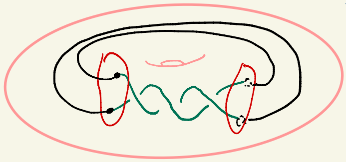
e.g.

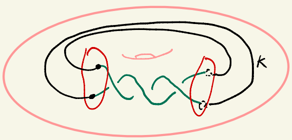


σ_1^3 (3 crossings)

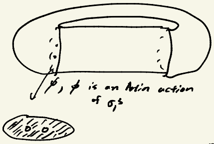
It's a trefoil knot.

idea 2: put it into a solid torus. (a mapping torus X ⇒ we know how to find $\pi_1(X)$ by using Van Kampen's Thm)





idea 3. find the fundamental group $\pi_1(T\setminus K)$



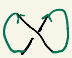
ϕ, ϕ is an Artin action of σ_i's

$$\pi_1(T\setminus K) = \langle \underbrace{\alpha_1, \dots, \alpha_n}_{\text{free}}, \overset{\text{meridian loop}}{\lambda} \mid \lambda\alpha_k\lambda^{-1} = \phi(\alpha_k) \rangle_{k=1,\dots,n}$$

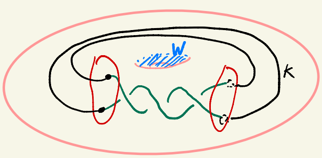
e.g.  $\phi(\alpha_1) = \alpha_1\alpha_2\alpha_1^{-1}$

$\phi(\alpha_2) = \alpha_1$

$$\Rightarrow \langle \alpha_1, \alpha_2, \lambda \mid \lambda\alpha_1\lambda^{-1} = \alpha_1\alpha_2\alpha_1^{-1}, \ \lambda\alpha_2\lambda^{-1} = \alpha_1 \rangle$$

 it's an unknot

idea 4. $\pi_1(T\setminus K)$ vs $\pi_1(\mathbb{R}^3\setminus K)$



$$\pi_1(\mathbb{R}^3\setminus K) \underset{\uparrow}{=} \pi_1((T\cup W)\setminus K)$$

Homotopy equivalent to $\mathbb{R}^3\setminus K$

By adding one more relation λ to cancel the bundary of the disk. W.

(Trefoil as $n=3$)

$$\pi_1(\mathbb{R}^3\setminus K) = \langle \alpha_1, \dots, \alpha_n \mid \alpha_1 = \phi(\alpha_1), \ \alpha_2 = \phi(\alpha_2), \ \dots, \ \alpha_n = \phi(\alpha_n) \rangle. \rightarrow (*)$$

take $\lambda = 1$

for trefoil: $n=3$, for each i, $\alpha_i = \phi(\alpha_i) = \alpha_i\alpha_j\alpha_i^{-1}$, $i\neq j$

in  K' example, $\pi_1(\mathbb{R}^3\setminus K')$ (Unknot)

$$= \langle \alpha_1, \alpha_2 \mid \alpha_2 = \alpha_1, \ \alpha_1 = \alpha_1\alpha_2\alpha_1^{-1} \rangle$$

$$= \langle \alpha_1, \alpha_2 \mid \alpha_2 = \alpha_1, \ \alpha_1 = \alpha_1 \rangle$$

$$= \langle \alpha_1 \rangle$$

$$\simeq \mathbb{Z}$$

i.e. 2-strand braid σ_1 (whose C_B = an unknot)

$\sigma_1(\alpha) = \alpha\beta\alpha^{-1}$ $\quad \begin{pmatrix} \alpha = \alpha_1 \\ \beta = \alpha_2 \end{pmatrix}$

$\sigma_1(\beta) = \alpha$

So, by the lemma

$$\langle \alpha, \beta, \lambda \mid \lambda\alpha\lambda^{-1} = \alpha\beta\alpha^{-1}, \ \lambda\beta\lambda^{-1} = \alpha \rangle$$

(Take $\lambda \to 1$)

$$= \langle \alpha, \beta \mid \alpha = \alpha\beta\alpha^{-1}, \ \alpha = \beta \rangle$$

$$= \langle \alpha \rangle$$

$$\simeq \mathbb{Z}$$

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