5.3.1: Calculate the value of C_n, for $1 \le n \le 8$.

By using the definition,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Thus,

$$C_1 = \frac{1}{1+1}\binom{2}{1} = 1$$

$$C_2 = \frac{1}{2+1}\binom{4}{2} = 2$$

$$C_3 = \frac{1}{3+1}\binom{6}{3} = 5$$

$$C_4 = \frac{1}{4+1}\binom{8}{4} = 14$$

$$C_5 = \frac{1}{5+1}\binom{10}{5} = 42$$

$$C_6 = \frac{1}{6+1}\binom{12}{6} = 132$$

$$C_7 = \frac{1}{7+1}\binom{14}{7} = 429$$

$$C_8 = \frac{1}{8+1}\binom{16}{8} = 1430$$

5.3.2:

1. Again, by using the definition,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

we have

$$(n+2)C_{n+1} = (4n+2)C_n.$$

2. Suppose for some n, C_n is a prime, and since we also know C_n can be rewritten in a recursive formula

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i},$$

so for all n which is larger than 3

$$\frac{(n+2)}{C_n} < 1,$$

that is

$$C_n > n+2.$$

3. So, $C_n \| C_{n+1}$, then let's assume $tC_n = C_{n+1}$, where t is a positive integer. Thus, $t = \frac{n+2}{4n+2}$, and this equation such that $k \in [1,3]$, and hence $n < 5$. Also, we know $C_4 = 14$ which is not a prime, so C_2, and C_3 are the only primes.

5.3.5:

Data collecting:
If $n = 1$, the only possibility for $a_1 = 1$.
If $n = 2$, the only possibilities are $(a_1, a_2) \in \{(1,1),(1,2)\}$
If $n = 3$, then $a_1 = 1$, a_2 could be 1, or 2.
Case 1: $a_2 = 1$
a_3 can be 1, 2, or 3.
$(a_1, a_2, a_3,) = (1,1,1)$, $(a_1, a_2, a_3,) = (1,1,2)$, or $(a_1, a_2, a_3,) = (1,1,3)$.
Case 2: $a_2 = 2$
$(a_1, a_2, a_3,) = (1,2,2)$, or $(a_1, a_2, a_3,) = (1,2,3)$
Conjecture (Claim):
The number of possibilities are Catalan numbers.
Proof. We can put these sequences into one-to-one correspondence with the paths discussed in the 5.3.5A that stay below diagonal.

In general, if we have a path P consisting of n horizontal direction steps and n vertical direction steps, we can look at the height of the kth horizontal direction step in the grid (which is equal to the number of vertical direction steps that precede the kth horizontal direction steps), and this is the a_k.
Therefore, we obtain $a_1 \le a_2 \le ... \le a_n$. Furthermore, the path P never crosses above the diagonal implies that $a_k \le k$ for $k = 1, 2, ..., n$.
Conversely, every sequence $1 \le a_1 \le a_2 \le ... \le a_n \le n$ with $a_1 \le 1, a_2 \le 2, ..., a_n \le n$ gives a path P that goes from $(0,0)$ to (n,n) without ever crossing above the diagonal line that joins $(0,0)$ to (n,n).

Therefore, the number of sequences $a_1, ..., a_n$ satisfying the stated conditions equals the number of paths from $(0,0)$ to (n,n) that never go above the diagonal, which we know is equal to C_n.

7.1.1:

$(1+x+x^2+x^3+x^4)^3 = 1+3x+6x^2+10x^3+15x^4+18x^5+19x^6+18x^7+...$
for $n<7$, I observed the recursive relation: $(a_n - a_{n-1}) - (a_{n-2} - a_{n-3}) = 1$
Also, as the exponent increases, the upper bound for the recursive relation increases.
Then, I assume

$a_0 = 1, a_1 = 3, a_2 = 6$
$a_n = a_{n-1} + a_{n-2} - an - 3 + 1$

To solve this recursion, and due to the knowledge of difference equation, I know this recursive relation is quadratic, so I assume $f(n) = an^2 + bn + c$
for $n = 0$, I got $c = 1$.
for $n = 1$, I got $a + b = 2$
for $n = 2$, I got $4a + 2b = 5$
Hence,

$$a = \frac{1}{2}, b = \frac{3}{2}.$$

$$f(n) = \frac{1}{2}n^2 + \frac{3}{2}n + 1,$$

and set $f(n) = a_n$.
Ans: $a_n = \frac{1}{2}n^2 + \frac{3}{2}n + 1, \forall n \leq 0$.

7.1.2:

Suppose $p(x) = ax^4 + bx^3 + cx^2 + dx + e$, then we have
LHS = $(ax^4 + bx^3 + cx^2 + dx + e)(x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + ...)$
RHS = $x^2 + x$
Firstly,
LHS=
$ex + 4ex^2 + 9ex^3 + 16ex^4 + 25ex^5 + ...$
$+dx^2 + 4dx^3 + 9dx^4 + 16dx^5 + ...$
$+cx^3 + 4cx^4 + 9cx^5 + ...$
$+bx^4 + 4bx^5 + ...$
$+ax^5 + ...$
Since LHS = RHS,
hence,
x^1: $ex = x$ so $e = 1$.
x^2: $(4e + d)x^2 = 1x^2$, so $d = -3$
x^3: $(9e + 4d + c)x^3 = 0x^3$, thus, $c = 3$
x^4: $(16e + 9d + 4c + b) = 0$, hence $b = -1$
x^5: $(25e + 16d + 9c + 4b + a) = 0$, hence $a = 0$.
Since the constraint (that is the RHS = $p(x)(\sum_{n=1}^{\infty}(x^n n^2))$) must be hold, so the upper bound of the degree of $p(x)$ is decided (and finite), and our trial $p(x)$
has hit the first zero ($a = 0$) coefficient of its highest degree term, so we know we've already hit the answer.

Ans: $p(x) = -x^3 + 3x^2 - 3x + 1$ or $-(x-1)^3$.

7.3.1:

(Based on the grid figure that used in the 7.3.1a proof).
Let X_n be the set of those sequence of n digits in which there are not consecutive 0's, 1's, and 2's.
Thus, $\#(X_n) = a_n$. And a sequence in X_n has either the form xa, where
$x \in \{3,4,5,6,7,8,9\}$, $a \in X_{n-1}$, or yxa, where
$y \in \{0,1,2\}$, $x \in \{3,4,5,6,7,8,9\}$, and $a \in X_{n-2}$.
There are $7a_{n-1}$ sequence of the first type and $3\ cdot 7 = 21$ sequences of the second type.
Therefore, the sequence $\{a_n\}$ satisfies the recurrence relation $a_n = 7a_{n-1} + 21a_{n-2}$.
The initial conditions are as follows:
we have $a_0 = 1$, because there is one sequence with zero digit in it, namely, the empty sequence, and this sequence doesn't include consecutive 0's, 1's, and 2's. Also, $a_1 = 10$, because each sequence consisting of a single digit that doesn't include consecutive 0's, 1's, or 2's digits.

$a_0 = 1, a_1 = 10$
$a_n = 7a_{n-1} + 21a_{n-2}$

7.4.1:

Set

$$A(x) = a_a x + a_2 x^2 + \sum_{n=3}^{\infty} a_k x^k$$

Since,

$$a_n = 3a_{n-1} + 4a_{n-2}, a_1 = 5, a_2 = 15$$

Hence,

$$\sum_{k=3}^{\infty} a_k x^k = 3\sum_{k=3}^{\infty} a_{k-1} x^k + 4\sum_{k=3}^{\infty} a_{k-2} x^k$$

Thus,

$$A(x) - a_1 x - a_2 x^2 = 3x(A(x) - a_1 x) + 4x^2(A(x))$$

$$A - 5x - 15x^2 = 3xA - 15x^2 + 4x^2 A$$

$$A = \frac{-5x}{4x^2 + 3x - 1}$$

$$= \frac{-5x}{(4x+1)(x-1)}$$

$$= \frac{B}{4x+1} + \frac{C}{x-1}$$

Solve:

$$[A(x)(4x+1)]_{x=\frac{-1}{4}} = B + C\left(\frac{4x+1}{x-1}\right)_{x=\frac{-1}{4}}$$

Thus, $B = -1$.

$$[A(x)(x-1)]_{x=1} = C + B\left(\frac{x-1}{4x+1}\right)_{x=1}$$

Thus, $C = -1$.

Make a double check:

$$\frac{-x+1-4x-1}{(4x+1)(x-1)}$$

which is okay!
It follows that

$$A(x) = \frac{-1}{4x+1} + \frac{-1}{x-1}$$

$$= (-1)\sum_{n=0}^{\infty}(-4x)^n + \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty}(-(-4)^n + 1)x^n$$

Make a triple check:

$$a_1 = 5, a_2 = 15$$

which is okay!

$a_n = 1 - (-4)^n, n \leq 0$

7.8.1:

$$a_n = a_{n-1} + \sum_{i=0}^{n-2} a_i a_{(n-2)-i}, n \geq 2, a_0 = a_1 = 1 \tag{0.1}$$

Data Collecting:
Only look the term $\sum_{i=0}^{n-2} a_i a_{(n-2)-i}$

$$n = 3, a_0 a_1 + a_1 a_0$$

which is generated by

$$(a_0 x^0 + a_1 x^1)(a_0 x^0 + a_1 x^1)$$

Then, if $n \to \infty$, and let $A(x) = \sum_{i=0}^{\infty} a_i x^i$

$$(A(x))^2 = (a_0 x^0 + a_1 x^1 + a_2 x^2 + ...)(a_0 x^0 + a_1 x^1 + ...) \tag{0.2}$$

Collecting terms that their coefficients have the same x^n factor:

$$x^0 : a_0 a_0$$
$$x^1 : a_0 a_1 + a_1 a_0$$
$$x^2 : a_0 a_2 + a_1 a_1 + a_2 a_0$$
$$.........$$

Also, we have (by substituting small numbers into the given recursion):

$$a_2 = a_1 + \sum_{i=0}^{0} a_0 a_0 = a_1 + a_0 a_0 = 2 \tag{0.3}$$

$$a_3 = a_2 + \sum_{i=0}^{1} a_0 a_{1-i} = a_2 + a_0 a_1 + a_1 a_0 = 4 \tag{0.4}$$

$$a_4 = a_3 + \sum_{i=0}^{2} a_0 a_{2-i} = a_3 + a_0 a_2 + a_1 a_1 + a_2 a_0 = 9 \tag{0.5}$$

$$.........$$

Derivation:
Suppose

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \tag{0.6}$$

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-1} + \sum_{n=2}^{\infty} \sum_{i=0}^{n-2} a_i a_{(n-2)-i} x^n \tag{0.7}$$

Substitute equation (0.6) into (0.7):

$$A(x) - x - 1 = (A(x) - 1)x + (A(x))^2 x^2 \tag{0.8}$$

$$A^2 x^2 + (x - 1)A + 1 = 0 \tag{0.9}$$

$$A(x) = \frac{-(x-1) - \sqrt{(x-1)^2 - 4x^2}}{2x^2} \tag{0.10}$$

Let's take the minus sign solution, so A could be positive,
since terms in series expansion are with a minus sign in front of them.

Simplified the $A(x)$

$$A(x) = \frac{-(x-1) - \sqrt{1-2x-3x^2}}{2x^2} \tag{0.11}$$

Taylor expansion at $x = 0$:

$$\sqrt{1-2x-3x^2} = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)(x^m)}{m!} \tag{0.12}$$

where

$$f^{(0)}(x) = \sqrt{1-2x-3x^2} \tag{0.13}$$

$$f^{(1)}(x) = \frac{-3x-1}{(1-2x-3x^2)^{1/2}} \tag{0.14}$$

$$f^{(2)}(x) = \frac{-4}{(1-2x-3x^2)^{3/2}} \tag{0.15}$$

$$f^{(3)}(x) = \frac{6(-6x-2)}{(1-2x-3x^2)^{5/2}} \tag{0.16}$$

$$f^{(4)}(x) = \frac{-48(9x^2+6x+2)}{(1-2x-3x^2)^{7/2}} \tag{0.17}$$

$$f^{(5)}(x) = \frac{-240(3x+1)(9x^2+6x+4)}{(1-2x-3x^2)^{9/2}} \tag{0.18}$$

$$\ldots\ldots.$$

$$\sqrt{1-2x-3x^2} = 1 - x - 2x^2 - 2x^3 - 4x^4 - 8x^5 - 18x^6 - 42x^7 - 102x^8 - \ldots \tag{0.19}$$

$$A(x) = \left(\frac{-(x-1)}{2x^2} - \frac{1}{2x^2} + \frac{x}{2x^2}\right) + \frac{2x^2}{2x^2} + \frac{2x^3}{2x^2} + \frac{4x^4}{2x^2} + \frac{8x^5}{2x^2} + \frac{18x^6}{2x^2} + \frac{40x^7}{2x^2} + \frac{102x^8}{2x^2} + \frac{254x^9}{2x^2} + \ldots \tag{0.20}$$

Let's cut to the chase:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + 2x^2 + 4x^3 + 9x^4 + 20x^5 + 51x^6 + 127x^7 + \ldots \tag{0.21}$$

The sequence of the coefficients are not in the on-line Encyclopedia of Integer Sequences.
Make a double check on the first four terms:
Comparing the coefficients of $A(x)$ to the given conditions:
$a_0 = 1$, which is fine.
$a_1 = 1$, which is okay.
Comparing the coefficients of $A(x)$ to equations (0.3), (0.4), and (0.5):
$a_2 = 2$, which is alright.
$a_3 = 4$, which looks good.
$a_4 = 5$, so it seems our result is correct!

In general, let $f(x) = \sqrt{1-2x-3x^2}$

$a_n = \frac{(-1)f^{(m)}(0)}{2(m!)}$, where $m = n+2$, $n \geq 0$