PROBLEM SOLVING

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Prove that

$$1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

Proof.
$n = 1 \Rightarrow 1 < 2$ which is true.
Suppose for $n = k$ we have

$$1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} < 2\sqrt{k}.$$

Then for $n = k + 1$ we have

$$1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$$

Need to show:

$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

If we multiply $\sqrt{k+1}$ for the both sides, and simplify the formula, we can show that the left-hand-side is $k^2 + k$, and right-hand-side is $k^2 + k + \frac{1}{4}$. This completes the proof. □

Prove that

$$2!4!...(2n)! \geq ((n+1)!)^n \tag{0.1}$$

Proof.
$n = 1 \Rightarrow 2! \geq (1 + 1)!$ which is true.
Suppose $n = k, k \in \mathbb{Z}$ is true.
Need to show:

$$2!4!...(2k)!(2k+2)! \geq ((k+1)!)^k(2k+2)!$$

l.h.s. of (0.1) as $n = k + 1$ is:

$$((k+1)!)^k 2(2k+2)!$$

r.h.s. of (0.1) as $n = k + 1$ is:

$$((k+2)!)^{k+1} = (k+2)!((k+2)!)^k = (k+2)!(k+2)^k((k+1)!)^k$$

Thus, after canceling the common term, we only need to show $(2k + 2)! > (k + 2)!(k + 2)^k$.
Since we can write the $(2k + 2)!$ into k-term of a product times a factorial $(k + 2)!$, and replace the k-term product with a lower bound $(k + 2)^k$ as follows

$(2k{+}2)! = (2k{+}2)(2k{+}1)...(k{+}3)(k{+}2)! > (k{+}2)^k(k{+}2)! =$ r.h.s. of what we need to show.

□

Enumerate the number of ways that we can tile a $2 \times n$ grid with 2×1 dominoes.

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Proof.
We can denote the number of ways to tile a $2 \times n$ grid with 2×1 dominoes as f_n. Then suppose we fill a 2×1 domino on its left end, then to tile the remaining $2 \times (n-1)$ grid, we have f_{n-1} ways. Secondly, if we tile a 2×2 grid of the $2 \times n$ grid on its left end, then there are f_{n-2} ways to tile the remaining grids. Thus, $f_n = f_{n-1} + f_{n-2}$. We also have the base cases: $f_1 = 1$, and $f_2 = 2$. It follows that it's a Fibonacci sequence, and it's general form is

$$f_n = \frac{\phi^n + \psi^n}{\sqrt{5}}$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}$$

and

$$\psi = \frac{1 - \sqrt{5}}{2} = 1 - \phi.$$

□