NUMBER THEORY HOMEWORK 3 (EC)

WILL

1. Conjectures and Proofs

Conjecture 1

Claim:
For the case $(6k+a)(6k+b)(6k+c) = (6k+d)(6k)(6k+e)$, there is no rational solutions for this equation, if $a, b, c, d, e \in \{1, 2, 3, 4, 5\}$.

Proof. Let's expand the equation, and simplify it:

$$(6k+a)(6k+b)(6k+c) = (6k+d)(6k)(6k+e)$$
$$\Rightarrow abc + (6ab + 6ac + 6bc - 6de)k + (36a + 36b - 36e)k^2 = 0$$
$$\Rightarrow \frac{abc}{(36a + 36b - 36e)} + \frac{(6ab + 6ac + 6bc - 6de)}{(36a + 36b - 36e)}k + k^2 = 0$$

Since $(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$, hence

$$\alpha\beta = \frac{abc}{36(a + b - e)} \in \mathbb{Z}$$
$$\alpha + \beta = \frac{-(ab + ac + bc - de)}{6 \cdot (a + b - e)} = \frac{de - a(b + c) - bc}{6(a + b - e)} \in \mathbb{Z}$$

where $k = \alpha$, or β, and $\alpha, \beta \in \mathbb{Z}$.
Therefore, we have two criteria for a, b, c, d, e to satisfy:

$$6 \mid de - a(b + c) - bc, (\exists \text{ symmetries: } b \longleftrightarrow c, d \longleftrightarrow e)$$
$$36 \mid abc.$$

Consider $36 = 2^2 \cdot 3^2$, and all the a, b, and e are belongs to the set $\{1, 2, 3, 4, 5\}$, meaning that the nonzero minimum of $\mid (a + b - e) \mid$ is $\mid (1+3-5) \mid = 1$. And even in this best case, we still don't have enough factors for 36. Hence,

$$36 \nmid abc,$$

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where

$$a, b, c \in \mathbb{Z}$$

.

$\square$

Conjecture 2

Claim:
For the case $(6k + a)(6k + b) = (6k + c)(6k + d)(6k)(6k + e)$, there is no rational solutions for this equation, if $a, b, c, d, e \in \{1, 2, 3, 4, 5\}$.

Proof. Consider

$$\begin{gathered}(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\\ = x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\alpha\gamma+\\ \alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta\end{gathered}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

Compare to our equation (after expanding, and simplifying):

$$k^4 + \frac{(216)(e+d+c)k^3 + 36(de+ce+cd-1)k^2 + 6(cde-b-a)k - ab}{1296} = 0$$

Since $k \in \mathbb{Z}$, hence

$$\alpha\beta\gamma\delta = \frac{-ab}{1296}$$

That is

$$1296 \mid ab \Rightarrow 1296 \leq ab,$$

Because $a, b \in \{1, 2, 3, 4, 5\}$, it follows that this is not possible. This completes the proof! $\square$

2. Appendix

Data collecting

- {2,3,4,5,6,7}
- {3,4,5,6,7,8}
- {4,5,6,7,8,9}
- {5,6,7,8,9,10}
- ....

Brutal force computing

If we assume the six consecutive number as:

$$\{6k, 6k+1, 6k+2, 6k+3, 6k+4, 6k+5\}$$

then the possible number of choices are governed by the Stirling number of second kind:

$$\left\{ {6 \atop 2} \right\} = 31.$$

Since we know that there are six cases can be excluded in the beginning, because it's not possible to single out one number, and equals to a product that is formed by the other five numbers.
Hence, we only need to consider the following 25 cases:

- $\{6k, 6k+1\}$, $\{6k+2, 6k+3, 6k+4, 6k+5\}$
  $\Rightarrow (6k)(6k+1) = (6k+2)(6k+3)(6k+4)(6k+5)$
  No rational solution.
- $\{6k, 6k+2\}$, $\{6k+1, 6k+3, 6k+4, 6k+5\} \Rightarrow (6k)(6k+2) = (6k+1)(6k+3)(6k+4)(6k+5)$
  No rational solution.
- $\{6k, 6k+3\}$, $\{6k+1, 6k+2, 6k+4, 6k+5\}$
  No rational solution.
- $\{6k, 6k+4\}$, $\{6k+1, 6k+2, 6k+3, 6k+5\}$
  No rational solution.
- $\{6k, 6k+5\}$, $\{6k+1, 6k+2, 6k+3, 6k+4\}$
  No rational solution.
- $\{6k+1, 6k+2\}$, $\{6k, 6k+3, 6k+4, 6k+5\}$
  No rational solution.
- $\{6k+1, 6k+3\}$, $\{6k, 6k+2, 6k+4, 6k+5\}$
  No rational solution.
- $\{6k+1, 6k+4\}$, $\{6k, 6k+2, 6k+3, 6k+5\}$
  No rational solution.

- $\{6k+1, 6k+5\}$, $\{6k, 6k+2, 6k+3, 6k+4\}$
  No rational solution.

- $\{6k+2, 6k+3\}$, $\{6k, 6k+1, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k+2, 6k+4\}$, $\{6k, 6k+1, 6k+3, 6k+5\}$
  No rational solution.

- $\{6k+2, 6k+5\}$, $\{6k, 6k+1, 6k+3, 6k+4\}$
  No rational solution.

- $\{6k+3, 6k+4\}$, $\{6k, 6k+1, 6k+2, 6k+5\}$
  No rational solution.

- $\{6k+3, 6k+5\}$, $\{6k, 6k+1, 6k+2, 6k+4\}$
  No rational solution.

- $\{6k+4, 6k+5\}$, $\{6k, 6k+1, 6k+2, 6k+3\}$
  No rational solution.

- $\{6k, 6k+1, 6k+2\}$, $\{6k+3, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k+1, 6k+2, 6k+3\}$, $\{6k, 6k+4, 6k+5\}$
  No rational solution.

- $\{6k+2, 6k+3, 6k+4\}$, $\{6k, 6k+1, 6k+5\}$
  No rational solution.

- $\{6k, 6k+2, 6k+4\}$, $\{6k+1, 6k+3, 6k+5\}$
  No rational solution.

- $\{6k, 6k+1, 6k+4\}$, $\{6k+2, 6k+3, 6k+5\}$
  No rational solution.

- $\{6k+1, 6k+2, 6k+4\}$, $\{6k+3, 6k+5, 6k\}$
  No rational solution.

- $\{6k+1, 6k+2, 6k+5\}$, $\{6k+3, 6k+4, 6k\}$
  No rational solution.

- $\{6k+1, 6k+3, 6k+4\}$, $\{6k+2, 6k+5, 6k\}$
  No rational solution.

- $\{6k+1, 6k+4, 6k+5\}$, $\{6k+2, 6k+3, 6k\}$
  No rational solution.

- $\{6k+2, 6k+4, 6k+5\}$, $\{6k+1, 6k+3, 6k\}$
  No rational solution.