NUMBER THEORY HOMEWORK 2: A CONJECTURE ON REPUNIT NUMBERS

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Data collecting

- $R_1 = 1.$
  $\rightarrow \gcd(R_1, R_1) = R_1.$

- $R_2 = R_1 \cdot 11.$
  $\rightarrow \gcd(R_2, R_2) = R_2.$
  $\rightarrow \gcd(R_2, R_1) = R_1.$

- $R_3 = 3 \cdot 37.$
  $\rightarrow \gcd(R_3, R_3) = R_3.$
  $\rightarrow \gcd(R_3, R_2) = R_1.$
  $\rightarrow \gcd(R_3, R_1) = R_1.$

- $R_4 = R_2 \cdot 101.$
  $\rightarrow \gcd(R_4, R_4) = R_4.$
  $\rightarrow \gcd(R_4, R_3) = R_1.$
  $\rightarrow \gcd(R_4, R_2) = R_2.$
  $\rightarrow \gcd(R_4, R_1) = R_1.$

- $R_5 = 41 \cdot 271.$
  $\rightarrow \gcd(R_5, R_5) = R_5.$
  $\rightarrow \gcd(R_5, R_4) = R_1.$
  $\rightarrow \gcd(R_5, R_3) = R_1.$
  $\rightarrow \gcd(R_5, R_2) = R_1.$
  $\rightarrow \gcd(R_5, R_1) = R_1.$

- $R_6 = R_2 \cdot R_3 \cdot 7 \cdot 13.$
  $\rightarrow \gcd(R_6, R_6) = R_6.$
  $\rightarrow \gcd(R_6, R_5) = R_1.$
  $\rightarrow \gcd(R_6, R_4) = R_2.$
  $\rightarrow \gcd(R_6, R_3) = R_3.$
  $\rightarrow \gcd(R_6, R_2) = R_2.$

$\rightarrow \gcd(R_6, R_1) = R_1$.

Conjecture 1 (If $n \mid m$.)

For example, $\gcd(R_6, R_3) = R_3$.

Claim: If $n \mid m$, then $R_n \mid R_m \Rightarrow \gcd(R_n, R_m) = R_{\gcd(n,m)}$.

In other words, suppose for all n as a divisor of m, the greatest common divisor of R_n and R_m is equal to R_n, and $n = \gcd(n, m)$.

Proof. Rewrite repunit number as a series:

$$R_k = \sum_{i=0}^{k-1} 10^i$$

Since $n \mid m$, so $\exists$ an integer a such that $m = an$.
Thus we have

$$R_m = R_{an} = \sum_{i=0}^{an-1} 10^i$$

which has an terms that starts from 10^0 to 10^{an-1}.
Since a and n are integers, so we can rearrange these finite an term into n columns and a rows such as

$$\sum_{i=0}^{an-1} 10^i$$

$$= 10^0 + 10^1 + 10^2 + ... + 10^{n-1}$$

$$+10^n + 10^{n+1} + 10^{n+2} + ...10^{2n-1}$$

$$+...$$

$$+10^{(a-1)n} + ... + 10^{an-1}$$

then we can use the first row to multiply the first term in each row to build all the original rows:

$$= (10^0 + 10^1 + 10^2 + ... + 10^{n-1}) \cdot (10^0 + 10^n + ... + 10^{(a-1)n}).$$

Why it's good to factor R_m as above? Because $10^0+10^1+10^2+...+10^{n-1}$ is R_n, and this completes the proof! $\square$

Summary

Since by combining the fact that $R_n \mid R_n$ with the above proved conjecture– "if $n \mid m$, then $R_n \mid R_m$", then by definition of the greatest common divisor, we have if $n \mid m$, then $\gcd(R_n, R_m) = R_n$.
Then since $n \mid n$, so combine the proof of the Conjecture 1, and by definition of the greatest common divisor, we obtain:
If $n \mid m$, $\gcd(n, m) = n$, then $\gcd(R_n, R_m) = R_n = R_{\gcd(n,m)}$.

Conjecture 2 (If $n \nmid m$.)

For instance, $\gcd(R_6, R_4) = R_2$.

Suppose $d = \gcd(n, m)$, if $n \nmid m$, then there are two cases:
(i) $d = 1$, and
(ii) $d \neq 1$.
Claim: Either way, if $d = 1$, or $d \neq 1$, $\gcd(R_n, R_m) = R_d$.

Proof. According to the above conjecture, there are two cases:

- Case 1. $d = 1$.

  Since $d = 1$, so $d \mid R_m$, and $d \mid R_n$.
  In the (visionary) proof in Conjecture 1, this means: for R_m we can only write the series in m rows with 1 column, which means each row only has one term, and each term is built by using $10^0 = 1$. Likewise, for R_n, we can only write the series in n rows with 1 column.

- Case 2. $d \neq 1$.

  By using the proof of the Conjecture 1, the series

  $$R_m = \sum_{i=0}^{m-1} 10^i$$

  could be rearranged into m/d rows, and d columns, and this could also be applied to R_n (i.e., n/d rows with d columns).

- According to the above two cases, for every $d = \gcd(m, n)$, $R_d \mid R_m$, and, $R_d \mid R_n$. Then by definition of greatest common divisor, $R_d = R_{\gcd(m,n)} = \gcd(R_n, R_m)$. This completes the proof!

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