AN ATTEMPT TO SOLVE (9,8,4,3,7)-LINKAGE PROBLEM

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Contents

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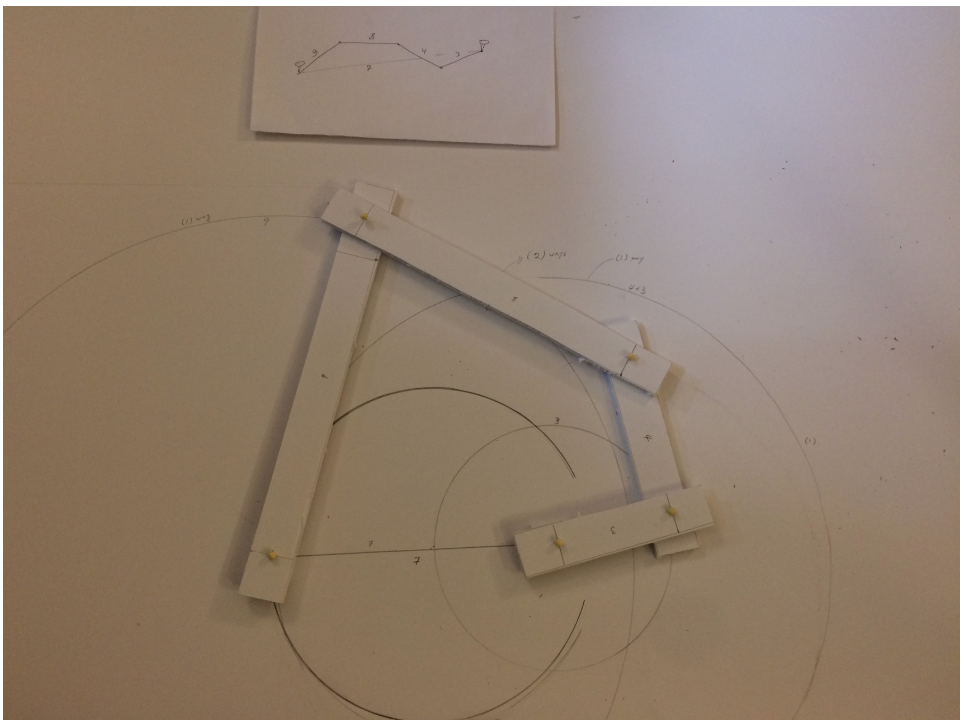
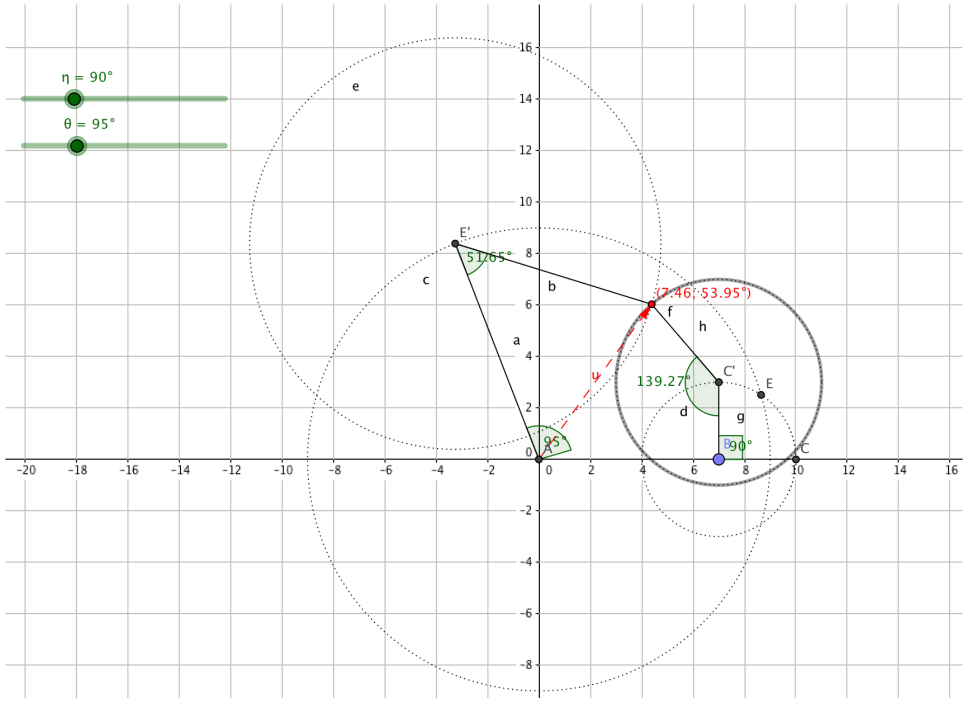


1. Set-up.

First of all, I made a model to investigate in details.

To be more precise, since the cardboard I used has a nonzero thickness, and each arm is not ideal line (has its nonzero width), so I want to see in more detailed. The way I do this is by using GeoGebra to draw geometry picture, and added a very simple code for controlling two arms that are linked to the ground arm with a slide bar. For instance:



Theoretically, each movement can be validated by using triangle inequality, but it's handy to check this just directly using a GeoGebra model to see the linkage moving in action continuously. A snapshot of this experiment is in the above figure.

2. Notations.

In the above GeoGebra code, I define the following symbols in Cartesian coordinate system:

Points.

- A is fixed at (0,0)
- B is fixed at (7,0)
- C's are the two intersections of the circle d and the x-axis
- C' is defined by the point B, and let B be the center of a circle d, with radius 3
- D is the intersection of two circles e and f
- E is the intersection of two circles c and d
- E' is the point on the circumstance of the circle c
- F is the point at (20, 0)

Circles.

- c is the circle centered at A with radius 9

- d is the circle centered at B with radius 3
- e is the circle centered at E' with radius 8
- f is the circle centered at C' with radius 4

Segments.

- a is the segment with endpoints A and E'
- b is the segment with endpoints E' and D
- g is the segment with endpoints B and C'
- h is the segment with endpoints C' and D
- i is the segment with endpoints C' and E'

Vector.

- u is a vector from point A to point D

Angles.

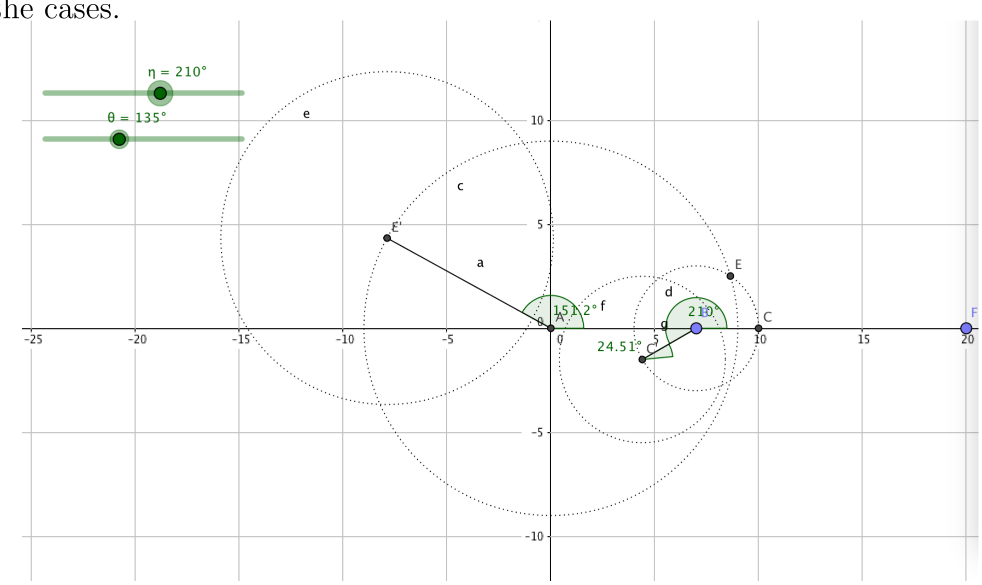
- α is the angle $\angle FAE'$
- β is the angle $\angle CBC'$
- γ is the angle $\angle BC'D$
- θ is $\alpha + 14.2°$, and a new freely moving variable for slide bar
- the value of η is exactly the same as β, and a new freely moving variable for slide bar

Terminology.

- The **topological space** M_l: it can also be understood as the **moduli space of planar n-gons**, viewed up to the action of orientation-preserving isometries of the plane. For any such planar n-gon, there is a unique rotation of the plane bringing the polygon in the position with a given side pointing in a fixed direction. Hence there is a one-to-one correspondence between the configuration space of the mechanism that if we fix one arm on the ground(3D)/plane(2D). So, M_l space is also called the **polygon space**, because it parameterizes shapes of all planar n-gons with sides $l_1, l_2, ..., l_n$. (*And, by considering knots as polygons, we can use this polygon space to study knot theory.*)

3. Fix length 3 bar to reduce the problem into a 4-bar linkage.

After experimenting for a while, I had a next idea: if we fixed the arm with length 3 at each position, then we can reduce our problem into (infinite many) 4-bar linkage problems. There is an advantage: if this arm is fixed instead of the other arm that is connected with the ground arm. The reason is the length 9 arm has a problem in a lot of the cases.



For example, as the above figure, if the angle CAE' is 151.2°, then we don't have a connected linkage. Hence with certain angles like this example, we don't have a connected linkage–which means we can't use an S^1 to glue all copies. And so, if we fix the shorter one (with length 3), then in each angle, we still can have a validated linkage that move in certain range of angles. (This will be proved in the next section.)

Then the advantage of fixing the length 3 arm is: we don't need to consider too much to be able to glue all the snapshots/slices of each 4-bar system (as a torus, but with some strips are cut out).

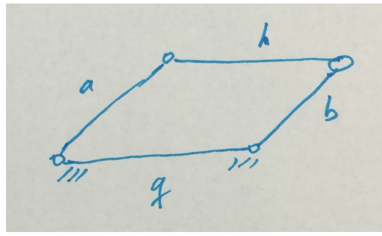
If we choose the arm with length 4 as the input bar in a 4-bar system, then all the possible angles that the linkage can maintain in connected is our domain. In this sense, the other arm with length 9 is our output bar (and the partial circumstance that is drew by the joint E' is the range of this mapping).

4. Classification of this four-arm linkage

The next step is we better show that for all possible positions of arm with length 3, by Grashof's theorem or its extension (according to wikipedia, and the reference therein[1])

Theorem 1. *Let's call the arm on the ground g, the arm on the top h, the bar on the left a, and the bar on the right b.*

We also followed the convention to define three quantities as follows:



$$T_1 = g + h - a - b$$
$$T_2 = b + g - a - h$$
$$T_3 = b + h - a - g$$

Table 1. Classification[1]

T_1	T_2	T_3	*Grashof condition*	*left bar*	*right bar*
negative	*negative*	*positive*	*Grashof*	*Crank*	*Crank*
positive	*positive*	*positive*	*Grashof*	*Crank*	*Rocker*
positive	*negative*	*negative*	*Grashof*	*Rocker*	*Crank*
negative	*positive*	*negative*	*Grashof*	*Rocker*	*Rocker*
negative	*negative*	*negative*	*Non-Grashof*	*0-Rocker*	*0-Rocker*
negative	*positive*	*positive*	*Non-Grashof*	π*-Rocker*	π*-Rocker*
positive	*negative*	*positive*	*Non-Grashof*	π*-Rocker*	*0-Rocker*
positive	*positive*	*negative*	*Non-Grashof*	*0-Rocker*	π*-Rocker*

Definitions:

- A crank: can rotate a full 360 degrees.
- A rocker: can rotate through a limited range of angles which does not include 0 degree or 180 degrees.
- A 0-rocker: can rotate through a limited range of angles which includes 0 degree but not 180 degrees.
- A π-rocker: can rotate through a limited range of angles which includes 180 degrees but not 0 degree.

We can show that what we claim in the previous section that if we the arm with length 3 is fixed at all possible positions, the remaining 4 bars can have a nonempty configuration space.

Now, suppose the length 3 bar, more precisely the point C', is arbitrarily fixed on the circumstance of the circle d. Hence, we got our 4-bar linkage. (That means we temporarily forget its existence, once we got a new given length of the ground arm.)

Except the ground, the other three arms with lengths: 9, 8, and 4.
It follows that the maximum of the possible ground arm of the four-bar linkage is 10, and the minimum is 7.

Hence, we consider two extreme cases:
Case 1: $h = 8, a = 9, b = 4,$ **and** $g = 10$

$$T_1 = 10 + 8 - 9 - 4 = 5 > 0$$

$$T_2 = 4 + 10 - 9 - 8 = -3 < 0$$

$$T_3 = 4 + 8 - 9 - 10 = -7 < 0$$

Thus, the bar on the left (a bar) is a **rocker**, and the right bar (b bar) is a **crank**.

Case 2: $h = 8, a = 9, b = 4,$ **and** $g = 4$

$$T_1 = 4 + 8 - 9 - 4 = -1 < 0$$

$$T_2 = 4 + 4 - 9 - 8 = -9 < 0$$

$$T_3 = 4 + 8 - 9 - 4 = -1 < 0$$

It follows that the bar on the left is a 0-**rocker**, and the right bar (b bar) is also a 0-**rocker**.

Case 3: when $T_1 = 0$, i.e., when $g = 5$

In this case, let's use Grashof's theorem, and it states that:

A four-bar mechanism has at least one revolving link if

$$s + l \leq p + q$$

And, all three mobile links will ***rock*** if

$$s + l > p + q$$

where s is the length of the shortest bar, l is the length of the longest bar, and p and q are lengths of intermediate bars.

In this case 3, we have $s = b = 4$, $l = a = 9$, and let $p = h = 8$, $q = g = 5$. It follows that:

$$s + l = 13$$

$$p + q = 13$$

so, we have at least one revolving link.

4.1. **Classified by Grashof's theorem.** The advantage of the above classification is it tells us which bar is a crank or a rocker. But the disadvantage is it doesn't work as $T_i = 0$. In this subsection I use Grashof's theorem to check the mobility of the link, and don't care about which link is a crank or rocker. If one uses this as a double check for the above case 1 and 2, one can see that the above classification is valid. Not only that since the lengths of a, b, h are fixed, we can plot them on R^1 space. Then consider the Grashof's theorem gives us an closed interval, $[4, 10]$, which is the set of all possible values of g. Hence, we have three cases:

- case a: $4 < g < 9$, so it won't be the s or l, and this is the case 3.
- case b: $g = 4$, so g can be the shortest, but since we already have one bar with length 4, so this case is reduced to case a.
- case c: $g > 9$, this will be checked in the following.

For$4 \leq g \leq 9$: we have $s = b = 4$, $l = a = 9$, and let $p = h = 8$, $4 \leq q = g \leq 9$.

$$s + l = 13$$

Sub-case (i) $5 \leq g \leq 9$

$$13 \leq p + q \leq 17$$

Hence, the 4-bar linkage has at least one revolving bar.

Sub-case (ii) $4 \leq g < 5$

$$12 \leq p + q < 13$$

Hence, all bars are rockers, and we get a double rocker mechanism.

For$g > 9$: we have $s = b = 4$, $l = a = 9$, and let $p = h = 8$, $q = g > 9$. It follows that:

$$s + l = 13$$
$$p + q > 17$$

Therefore

$$s + l < p + q$$

Then, according to the theorem, the 4-bar linkage has at least one revolving bar!!

4.2. **A quick summary.** In the above two ways of classifications, we see that as long as the four bars are linked to each other, then according to the theorem we only have two types of machines: double cranks or double rockers. **That means each mobile joint can be moved for a certain amount of length, and for our purpose, this gives us a non-empty configuration space.** Hence, my next strategy is to decide which bar should be fixed to get linked linkage in all the cases. The answer is if we fixed the bar with length 9, then for example as $\theta = 170°$, in GeoGebra, we can see that we don't have a linkage, no matter how we rotate the shortest bar (because two circle don't intersect in this case). Hence, we should fix the shortest bar, and find when we can get a linkage when we rotate the bar with length 9 (This can bring us more advantages in analyzing its configuration space.)

Thus, if we see the shortest bar as the controlled parameter (or the so-called free parameter), then we can consider the configuration space as people did in gauge theory that, along with the track of the free parameter, each value of the free parameter is corresponding to a torus. Here, so far, we know in our problem, our track of the controlled parameter is an S^1 which is generated by the length 3 bar rotating in 360 degrees freely. And analogue to torus, we have 2 to 4 curve segments generated by the two joints (between 8 and 9, and 9 and 4). It's not possible to draw infinite many snapshots or take those many screen shots of the configuration spaces of the two joints for each position of the length 3 bar. Nonetheless, I analyze them by considering $n \cdot 45°$ where n is an integer.

5. Detailed analysis of 8 cases

In the following I only drew one direction.

Strategy

- Step 1: fix the length 3 bar at a certain angle
- Step 2: rotate the length 8 bar in the GeGebra form 0 to 360 degrees
- Step 3: mark the traces of the two joints with two colors: *red* and *green*

The following are some important remarks for counting the number of copies of the traces.

- **It's important to know that if we rotate the bar with length 9 in the model I made for this problem in GeoGebra, then each pattern of all five bars are unique.**
  Because after we fix the shortest bar, the number of remaining joints that can move within the constraint are two.

- ⇒**So, we only focus on counting how many copies of traces on two circles (c and f). And, in the end, we can use the S^1 of the trace of the bar with length 3 to glue each copy into a strip (it will be like a torus that was cut out a strip, and contain another torus inside, and also got cut out a strip, and two surfaces are closed (with boundary) and intersect to each other).**

- **Although each pattern is unique, the traces on the circle c and f can be overlap. To avoid missing anything, I use the orientation when the bars are rotated, because if the joint is reach its turning point, then the following trace will be in an opposite orientation, i.e., it starts to overlap the previous trace.**

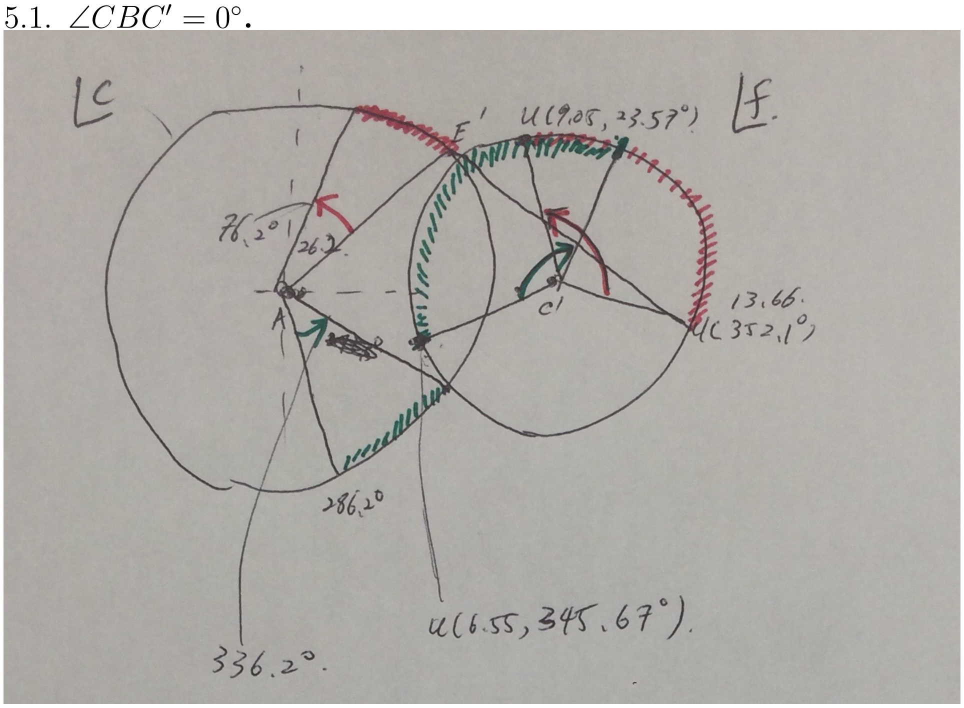
- **However, since we have two moving joints, and for each joint by symmetry we can have four copies that are generated by flipping the other two jointly bars about the diagonal axis.**
  ⇒Therefore we have to multiply the result by four in the end.

In all cases, when we start off the degree zero of the bar with length 8, the joint of this bar on the circle c **rotate counter-clockwise, and the joint between the bar with the length 9 and the bar with length 4 is also rotating counter-clockwise. Let's mark them *in pair* (since they have the same orientation) with *red*.**

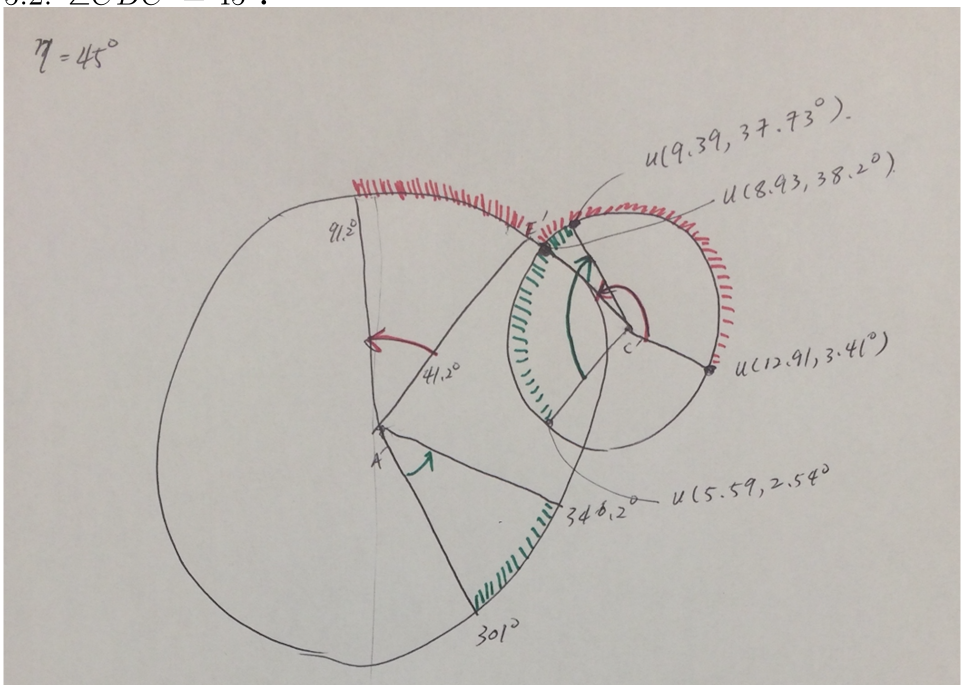
Then, let's keep increasing the angle till the 4 bar system got disconnected and connected again, so once the bar with length 8 reach a certain point, the length bar with length 4 will start to rotate clockwise. Let's mark the traces in green in a pair.

The following are the snapshots of those colored configurations on the **circle c** and **circle f**.

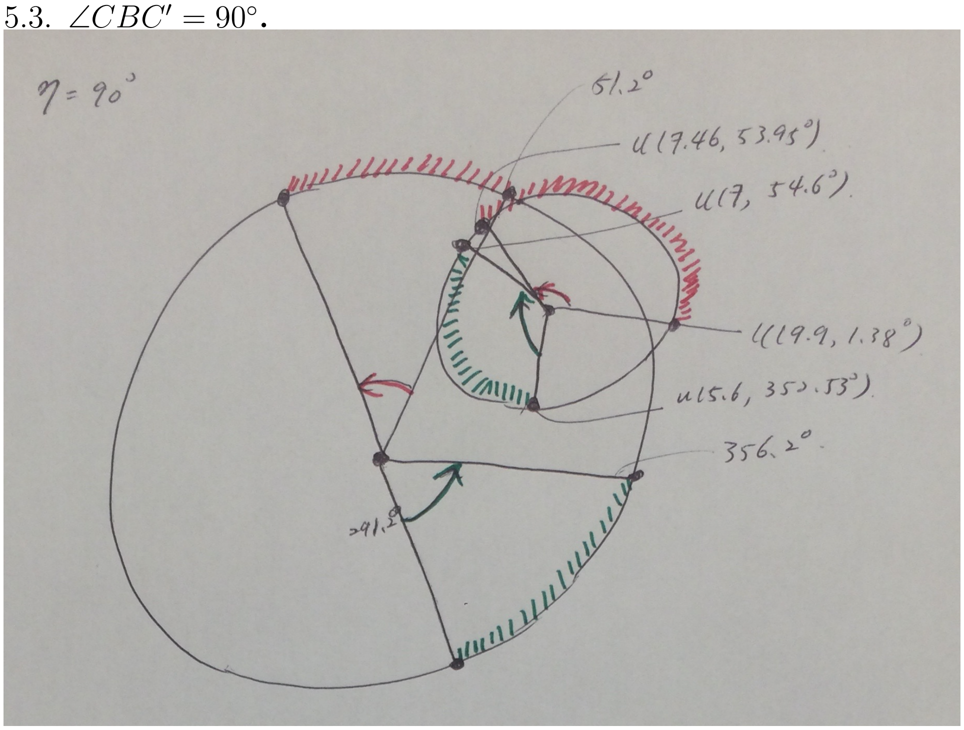
5.1. $\angle CBC' = 0°$.



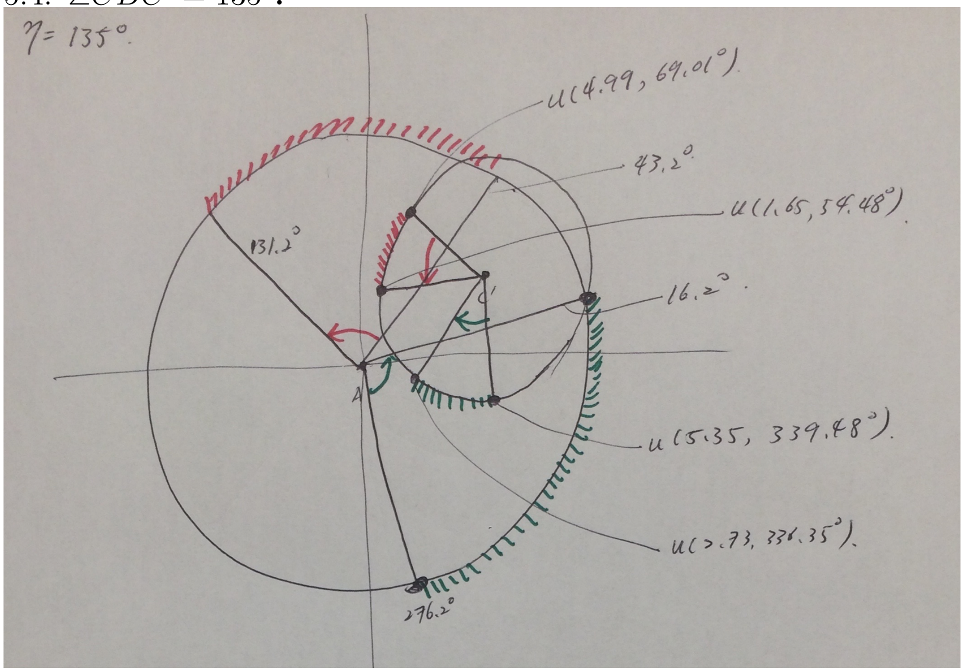
5.2. $\angle CBC' = 45°$.



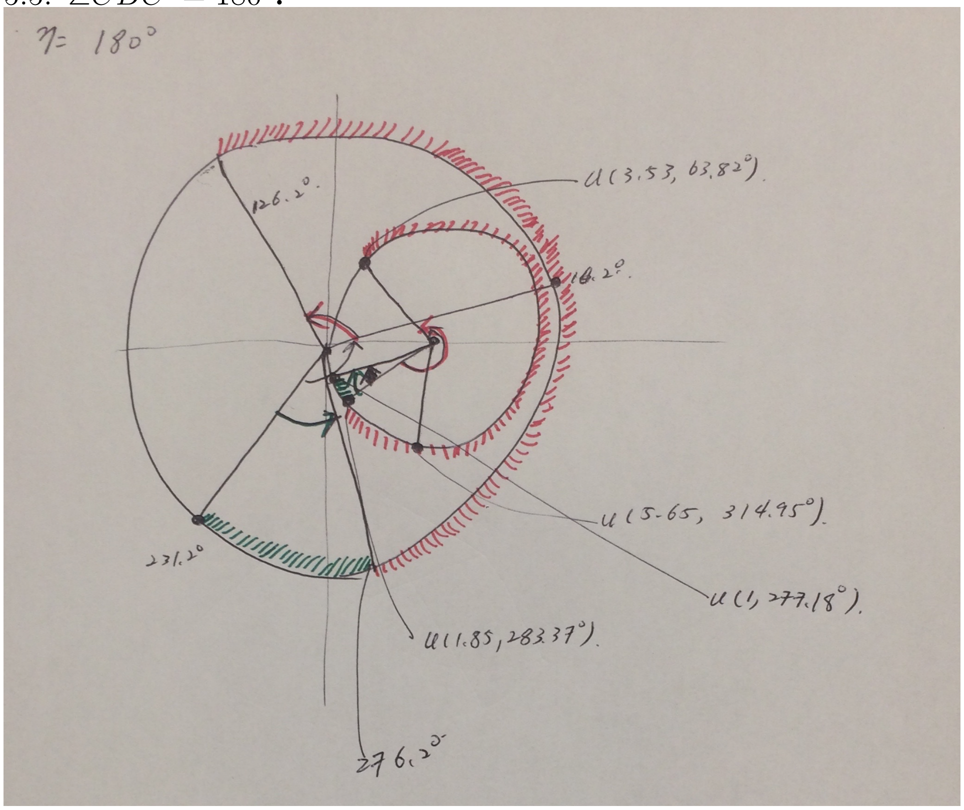
5.3. $\angle CBC' = 90°$.



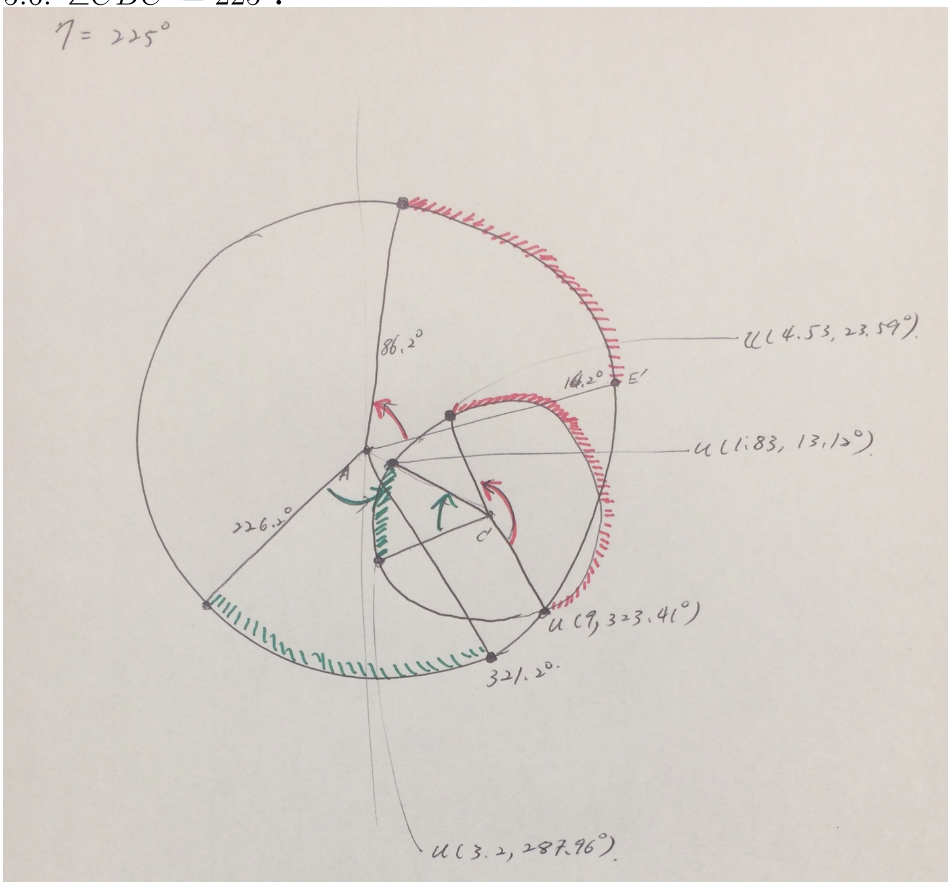
5.4. $\angle CBC' = 135°$.



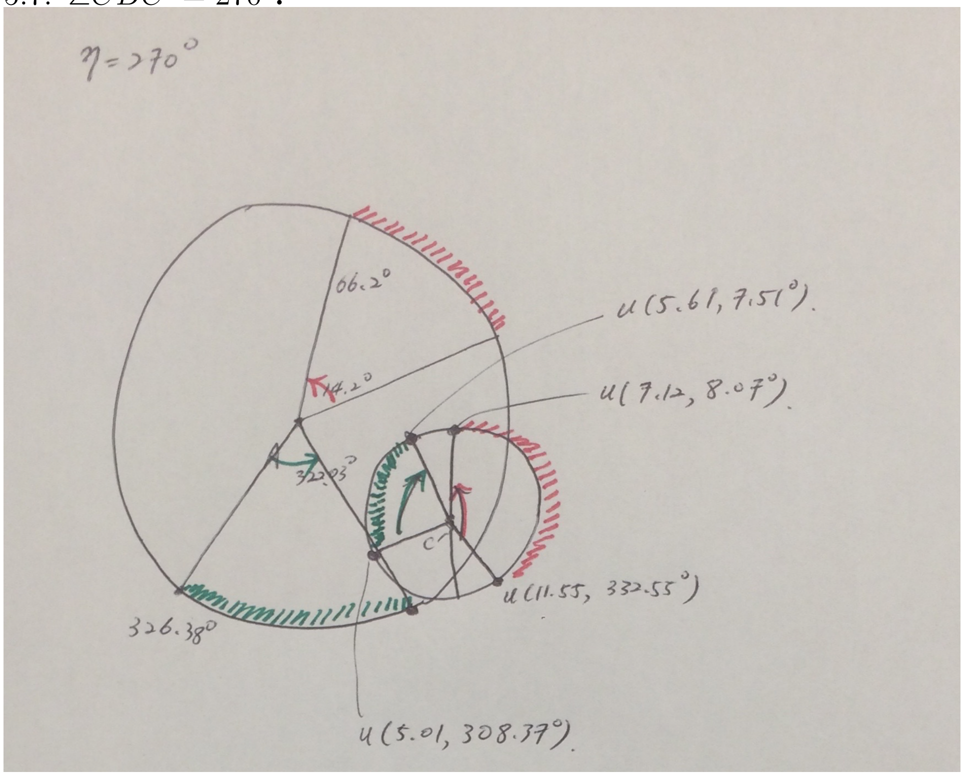
5.5. $\angle CBC' = 180°$.



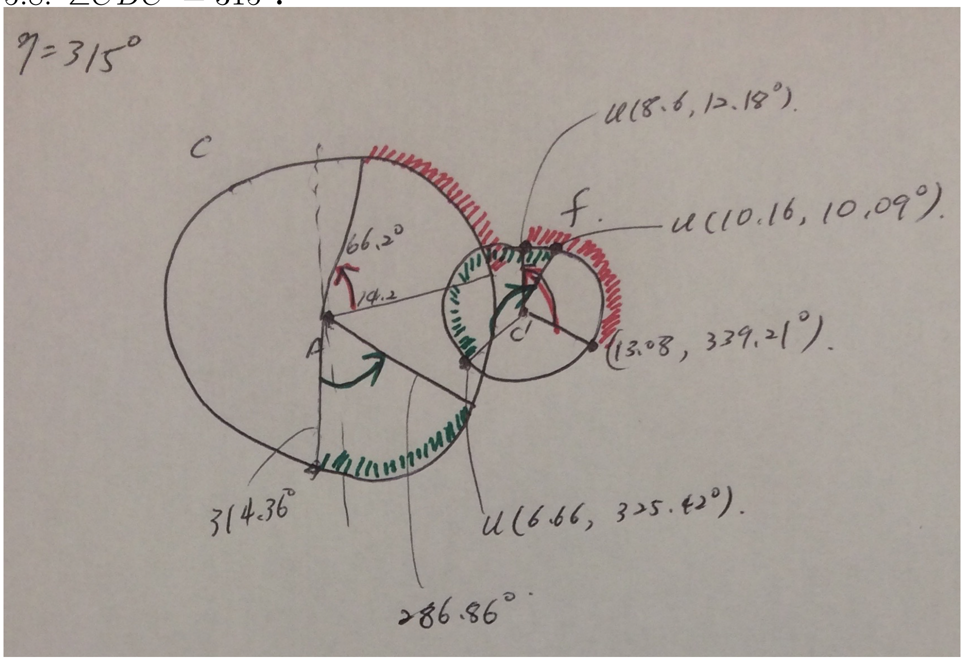
5.6. $\angle CBC' = 225°$.



5.7. $\angle CBC' = 270°$.



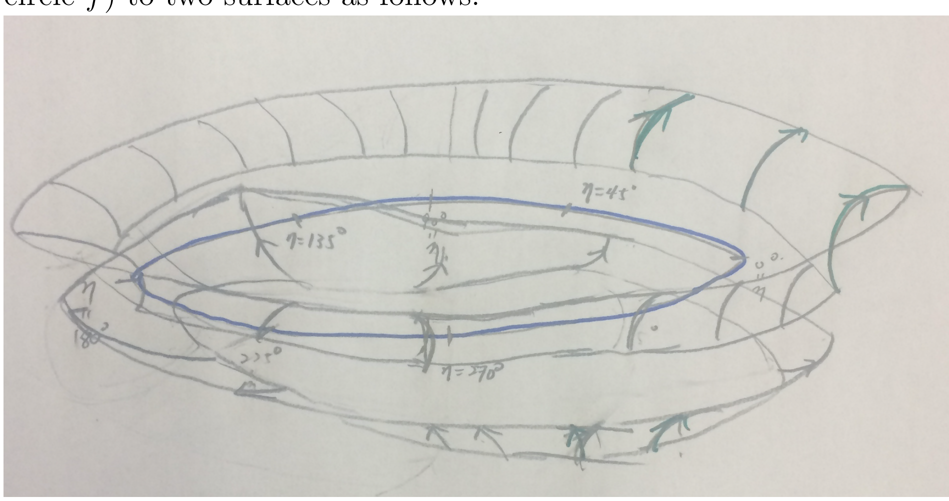
5.8. $\angle CBC' = 315°$.



6. Gluing

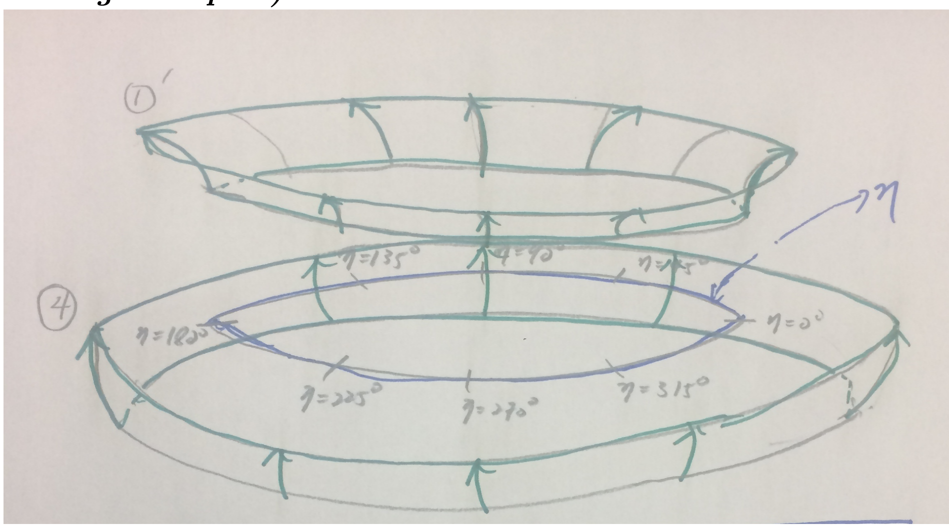
First of all, when $\eta = 0°$, $\eta = 45°$, and $\eta = 315°$ of the above examples, on circle f, traces are overlapping. So, I choose to make the green segment has one endpoint missing (isomorphic to $[0, 1)$), and the red segment is isomorphic to $[a, b], a > 1$. Hence, we can glue two segments into one segment that is isomorphic to $[0, 1]$.

Secondly, let's glue all green segments (including the red segment on circle f) to two surfaces as follows:



Note the blue circle is the trace made by the bar with length 3. So, geometrically, we can use this circle to parametrize the two 2d surfaces. This gluing is similar to the integration of a symmetric 2d surface by using a line segment–suppose the traces parametrized by the zero degree of the length 3 bar is on x-y plane, then we swiping the two green curve segments from zero degree to 360 degree about the y-axis.

Since all we care is the topology, hence we can make it to be more smooth ***(the following figure is the partial result without the red segment part)***:



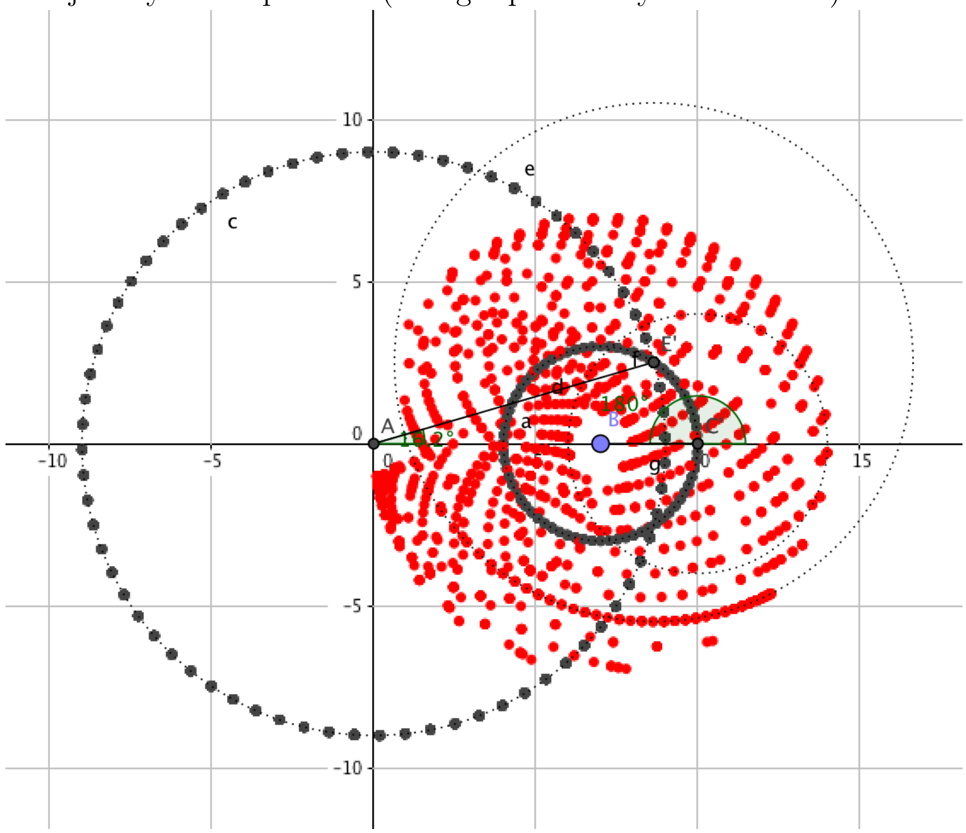
Likewise, for all the red segments on the circle c, we do the same. It's important to notice that as $\eta = 315°$ (as a slice view), we can see that there are two surfaces intersect to each other. Because of this, I don't know the exact name of this structure.

Finally, we multiply the result with a factor 4, since we have four copies.

7. Discussion

Problem 1: If we don't think in this way, say if we fix the bar with length 8, and move the bar with length 3, we got the trouble that for the endpoint of the bar with length 8, at each point (I have shown that in the Classification section) we can move the bar with length 3 in a certain angle without breaking the linkage. However, this can imply the number of copies at the endpoint of the bar with length 8 is infinite.

Problem 2: It's not like (1,1,1,1,1)-linkage, in this case the joint (like point D) also break into the circle made by the bar that has one end pinned on the plane. To show this, in the following figure, I used GeoGebra to trace the trajectory of the three joints. The red dots are the trajectory of the point D (also get pointed by the vector u).



Red dots only can be drawn as the linkage is not disconnected. We can notice that the black dots invade the region that originally protected by the two bars that are pinned on the plane. In the (1,1,1,1,1) case, if we take the horizontal slice of the M_2, we can see the shape of the boundary of the trajectories of the joints. But in this (8,9,4,3,7) case, the shape is quite complicated, and it's hard to count how many copies of each point have. **Solution:** The same solution for the two above problems is we reduce the 5-bar system in two 4-bar. We take advantage to the trajectory of the bar with length 3. Because that trajectory is a complete S^1! So, each time, we only look at each 4-bar linkage with the length of the ground bar vary. Then we accumulate all the configuration subspaces in those slices, and glue them together in to a surface which is parametrized by the S^1 of the bar with length 3.

8. Conclusion

Although the name of this structure is not in my knowledge, if we only think of the complete circle c and f, and use S^1 (the complete circle of the trace of the endpoint of the arm with length 3), then we can derive two complete tori.

In this given problem, if we look at the all the colored traces on circle c, and think of they are sliced from a 2d surface, then we can see the there is a open shell (meaning that it is connected from $\eta = 0°$ to $\eta = 360°$) got cut out. And one disk also got removed (it's a disk instead of a shell), because when $\eta = 180°$, we can see that the green and red segment are connected.

Hence, the configuration space is made of two surfaces:

- the one that is generated by the circle c got cut out one open shell and one open disk, but the surface is closed with boundary
- the one that is generated by the circle f also got cut out one open shell and one open disk, but the surface is also closed with boundary
- the two about surfaces intersect to each other.
- we have 4 copies of the above intersecting surfaces

9. References.

[1] J. M. McCarthy and G. S. Soh, Geometric Design of Linkages, 2nd Edition, Springer, 2010.